XV. *On a Class of Differential Equations, including those which occur in Dynamical Problems.*—Part II. *By* W. F. Donkin, *M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy in the University of Oxford.*

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THE following paper forms the continuation and conclusion of one on the same subject presented to the Royal Society last year, and printed in the Philosophical Transactions for 1854. I have however put it, as far as possible, in such a form as to be independently intelligible.

The fourth Section (the first of this Part) contains a recapitulation of some of the most important results of the former Part, in the form of seven theorems, here enunciated without demonstration.

In the fifth Section the method of the variation of elements is treated under that aspect which belongs to it in connexion with the general subject. It is applied, by way of example, to deduce the expressions for the variations of the elliptic elements of a planet's orbit from the results of art. 30 (Part I.), on undisturbed elliptic motion; this example was chosen, partly because the resulting expressions are required in a future section, and partly for the sake of incidentally calling attention to a fallacy which has been, perhaps, often committed, and certainly seldom noticed. The same method, under a slightly different and possibly new point of view, is applied, as a second example, to the problem of the motion of a free simple pendulum, omitting the effect of the earth's rotation. I believe the methods of this paper might be advantageously employed in the treatment of that general form of the problem of a free pendulum which has been considered by Professor Hansen in his Prize Essay. I was unwilling, however, to attempt what might have turned out to be merely an unconscious plagiarism, without having seen the Essay in question, of which I only succeeded in obtaining a copy on the day of writing this preface. As I now perceive that the investigation would be quite independent, I hope to enter upon it at some future time.

The sixth Section contains some general theorems concerning the transformation of systems of differential equations of the form considered in this paper, by the substitution of new variables. The most important case consists in the transformation from fixed to moving axes of coordinates, in dynamical problems. Some of the results are, I think, interesting, and perhaps new.

The seventh and last Section contains an application of the preceding theorems, in connexion with the variation of elements, to the transformation of the differential equations of the planetary theory. This investigation, if interesting at all, will probably be so to the mathematician rather than to the astronomer. I think, however, that if the theories of physical astronomy were more frequently treated rigorously and symmetrically, apart from any approximate integrations; and if, when the latter are introduced, more care were taken to give a clear and exact view of the nature of the reasoning employed, it might be possible to draw the attention and secure the cooperation of a class of mathematicians who now may well be excused, if, after a slight trial, they turn from the subject in disgust, and prefer to expatiate in those beautiful fields of speculation which are offered to them by other branches of modern geometry and analysis.

The contents of the two last Sections are more or less closely connected with the subjects of various memoirs by other writers, especially Professor HANSEN and the Rev. B. BRONWIN. I cannot pretend to that degree of acquaintance with them which would enable me to give an exact statement of the amount of novelty to be found in my own researches. I believe it is enough to justify me in offering them to the Society; beyond this I make no claim.

Oxford, Feb. 15, 1855.

SECTION IV.

49. The following theorems were demonstrated in the former part of this essay, and are recapitulated here for convenience of reference. (As before, total differentiation with respect to the independent variable t will, in general, be denoted by accents, which will be used *for no other purpose.*)

Theorem I.—If X be a function of n variables $x_1, x_2, \dots x_n$, and if $y_1, y_2, \dots y_n$ be n other variables connected with the former by the n equations

$$\frac{dX}{dx_1}=y_1, \frac{dX}{dx_2}=y_2, \dots \frac{dX}{dx_n}=y_n, \quad \dots \dots \quad (50.)$$

then will the values of $x_1, x_2, \dots x_n$, expressed by means of these equations in terms of $y_1, \dots y_n$, be of the form

$$x_1=\frac{dY}{dy_1}, x_2=\frac{dY}{dy_2}, \dots x_n=\frac{dY}{dy_n}; \quad \dots \dots \quad (51.)$$

and if p be any other quantity explicitly contained in X, then also

$$\frac{dX}{dp}+\frac{dY}{dp}=0 \quad \dots \dots \quad (52.)$$

(the differentiation with respect to p being in each case performed only so far as p appears *explicitly* in the function).

The value of Y is given by the equation

$$Y=-(X)+(x_1)y_1+(x_2)y_2+\dots+(x_n)y_n, \quad \dots \dots \quad (53.)$$

where the brackets indicate that $x_1 \dots x_n$ are supposed to be expressed in terms of $y_1 \dots y_n$ (arts. 2, 3.).

Theorem II.—Suppose the function X to contain explicitly, besides the n variables $x_1 \dots x_n$, another variable t, and also n constants $a_1, a_2, \dots a_n$; and in addition to the equations (50.), let the following be assumed:

$$\frac{dX}{da_1}=b_1, \dots \frac{dX}{da_n}=b_n, \quad \quad (54.)$$

where $b_1, \dots b_n$ are n other constants; so that, by virtue of the $2n$ equations (50.), (54.), the $2n$ variables $x_1 \dots x_n, y_1 \dots y_n$, may be considered as functions of the $2n$ constants $a_1, \dots a_n, b_1, \dots b_n$, and t. Then if from the equations (50.), (54.), and their total differential coefficients with respect to t, the $2n$ constants be eliminated, there will result the following $2n$ simultaneous differential equations of the first order; viz.—

$$x_i'=\frac{dZ}{dy_i}, \; y_i'=-\frac{dZ}{dx_i}, \quad \quad (55.)$$

where Z is a function of $x_1 \dots x_n, y_1, \dots y_n$ (which will in general also contain t explicitly), and is given by the equation

$$Z=-\left(\frac{dX}{dt}\right). \quad \quad (56.)$$

In this equation $\frac{dX}{dt}$ represents the partial differential coefficient of X taken with respect to t so far as t appears explicitly in the original expression for X in terms of $x_1 \dots x_n, a_1 \dots a_n$ and t; and the brackets indicate that $a_1, \dots a_n$ are afterwards to be expressed in terms of the variables by means of the equations (50.), (arts. 5, 6.)

Theorem III.—Let the supposition that the $2n$ variables $x_1 \dots x_n, y_1 \dots y_n$ are expressed in terms of the $2n$ constants and t, be called *Hypothesis* I.; and the converse supposition that $a_1 \dots a_n, b_1 \dots b_n$ are expressed in terms of the $2n$ variables and t, *Hypothesis* II.; then will the following relations subsist:

$$\left.\begin{aligned} \frac{dx_i}{da_j}&=-\frac{db_j}{dy_i}, \; \frac{dx_i}{db_j}=\frac{da_j}{dy_i} \\ \frac{dy_i}{da_j}&=\frac{db_j}{dx_i}, \; \frac{dy_i}{db_j}=-\frac{da_j}{dx_i} \end{aligned}\right\} \quad \quad (57.)$$

(In each of these equations the first member refers to *Hyp.* I., and the second to *Hyp.* II.; and since there is no connexion between the indices of the variables and those of the constants, the case of $i=j$ has no peculiarity.)

Theorem IV.—Let the symbol $[p, q]$ be an abbreviation for the expression

$$\Sigma_i\left(\frac{dp}{dy_i}\frac{dq}{dx_i}-\frac{dp}{dx_i}\frac{dq}{dy_i}\right)$$

(where p, q are any functions of the $2n$ variables, which may also contain any other quantities explicitly; and the differentiations are performed only so far as x_1,&c.,y_1,&c. appear explicitly in p, q); then if $a_1, \dots a_n, b_1, \dots b_n$ be expressed (*Hyp.* II.) in terms of the $2n$ variables and t, the following equations subsist identically:

$$[a_i, b_i]=-[b_i, a_i]=1, \; [a_i, b_j]=[a_i, a_j]=[b_i, b_j]=0 \quad \quad (58.)$$

(i being different from j); and obviously in all cases

$$[p, q]=-[q, p], \text{ and } [p, p]=0 \quad \text{(art. 9.)}.$$

Theorem V.—If u, v be either (1) any two functions whatever of the $2n$ constants a_1, &c., b_1, &c., or (2) any two functions whatever of the $2n$ variables x_1, &c., y_1, &c. (which may in either case also contain t explicitly)*, then

$$\Sigma_i\left\{\frac{du}{dy_i}\frac{dv}{dx_i}-\frac{du}{dx_i}\frac{dv}{dy_i}\right\}=\Sigma_i\left\{\frac{du}{da_i}\frac{dv}{db_i}-\frac{du}{db_i}\frac{dv}{da_i}\right\}. \quad \text{(59.)}$$

(When u, v represent functions of the constants, the differential coefficients in the first member of this equation refer to *Hyp.* II.; and, when functions of the variables, those in the second member refer to *Hyp.* I.) (art. 10.).

Theorem VI.—Let $x_1, \ldots x_n, y_1, \ldots y_n$ be $2n$ variables concerning which no supposition is made except that they are connected by n equations of the form

$$a_i=\phi_i(x_1, x_2, \ldots. x_n, y_1, y_2, \ldots y_n) \quad \text{(a.)}$$

(where the functions on the right are only subject to the condition that the n equations (a.) shall be algebraically sufficient to determine $y_1, \ldots y_n$ in terms of $x_1, \ldots x_n$, a_1, &c., and may contain explicitly any other quantities besides x_1, &c., y_1, &c.).

Then, if by means of the equations (a.) the n variables $y_1, y_2, \ldots y_n$ be expressed as functions of x_1, x_2, &c., a_1, &c.; in order that the $\frac{n(n-1)}{2}$ conditions

$$\frac{dy_i}{dx_j}=\frac{dy_j}{dx_i}$$

may subsist identically, it is necessary and sufficient that each of the $\frac{n(n-1)}{2}$ expressions $[a_i, a_j]$ vanish identically.

Theorem VII.—Let Z be any function whatever of $2n$ variables $x_1 \ldots x_n$, $y_1 \ldots y_n$, and t. If of the system of $2n$ simultaneous differential equations of the first order

$$x'_i=\frac{dZ}{dy_i},\ y'_i=-\frac{dZ}{dx_i} \quad \text{(I.)}$$

there be given n integrals involving n arbitrary constants $a_1, a_2, \ldots a_n$, so that each of these constants may be expressed as a function of the variables x_1, &c., y_1, &c. (with or without t); then *if the* $\frac{n(n-1)}{2}$ *conditions* $[a_i, a_j]=0$ *subsist identically, the remaining* n *integrals may be found,* as follows. By means of the n given integrals let the n variables $y_1 \ldots y_n$ be expressed in terms of x_1, &c., a_1, &c.; and let (Z) be what Z becomes when $y_1 \ldots y_n$ are thus expressed. These values of $y_1, y_2 \ldots y_n$ and $-$(Z), will be the partial differential coefficients with respect to $x_1, x_2, \ldots x_n$ and t, of one and the same function; call this function X, then, since its partial differential coefficients are

* It was inadvertently stated in art. 10, that u, v must *not* contain t explicitly. But it is evident that no such limitation is implied in the demonstration of the theorem. The preceding theorem is obviously a particular case of this; namely, the case in which $u=a_j$, $v=b_j$.

all given $\left(\text{by the equations } \frac{d\mathrm{X}}{dx_i}=y_i, \frac{d\mathrm{X}}{dt}=-(\mathrm{Z})\right)$, X may be found by simple integration, and is therefore to be considered a given function of $x_1, \ldots x_n, a_1, \ldots a_n$ and t. The remaining n integrals are then given by the n equations

$$\frac{d\mathrm{X}}{da_i}=b_i,$$

$b_1 \ldots b_n$ being n new arbitrary constants.

[On the relation between this theorem and the theories of Sir W. R. Hamilton and Jacobi, see arts. 15–20.]

50. Other results established in the former part will be referred to as occasion may require. To the theorems enunciated in the preceding article, the following may now be added.

Returning to the equations (50.), (54.), (55.), we may observe, that if, in the *first* members of (55.), x_i, y_i be supposed expressed in terms of a_1, &c., b_1, &c. and t, then $\frac{dx_i}{dt}, \frac{dy_i}{dt}$ may be written instead of x_i', y_i'; since on this hypothesis the total differential coefficients of x_i, y_i are obtained by differentiating with respect to t as it appears explicitly. We have therefore

$$\frac{dx_i}{dt}=\frac{d\mathrm{Z}}{dy_i}, \quad \frac{dy_i}{dt}=-\frac{d\mathrm{Z}}{dx_i},$$

where the first members refer to *Hyp.* I., and the second to *Hyp.* II. But since the equations (50.), (54.) involve a, b, exactly in the same way as they involve x, y, it is obvious that the same reasoning which leads to the equations just written, would lead, *mutatis mutandis*, to the following, which may be considered as an addition to the system of equations (57.) (Theorem III.):

$$\frac{da_i}{dt}=\frac{d\mathrm{Z}}{db_i}, \quad \frac{db_i}{dt}=-\frac{d\mathrm{Z}}{da_i}. \qquad (60.)$$

In these equations, a_i, b_i in the *first* members are supposed to be expressed in terms of the variables (*Hyp.* II.), whilst in the second members x_1, &c., y_1, &c. are supposed to be expressed in terms of the constants and t (*Hyp.* I.). As before, $\mathrm{Z}=-\frac{d\mathrm{X}}{dt}$, but in (60.) Z is differently expressed, being what the Z of (55.) becomes when x_1, &c., y_1, &c. are expressed according to *Hyp.* I.

It is to be remembered that all consequences deduced from the form of the system (50.), (54.) belong to the system of equations, obtained as in Theorem VII., which express the solution of the differential equations (I.). Such a solution will be called, as before, a *normal* solution; and the system of equations obtained by expressing a_1, &c., b_1, &c. in terms of the variables and t, will be called a system of *normal integrals*. (See art. 20, and the note to art. 29.)

51. Let a_1, &c., b_1, &c. be called, as before, *elements*. If then c be any function of the elements, when the latter are expressed in terms of the variables and t

(*Hyp.* II.), c becomes also a function of the same; and we have

$$\frac{dc}{dt}=\Sigma_i\left(\frac{dc}{da_i}\frac{da_i}{dt}+\frac{dc}{db_i}\frac{db_i}{dt}\right)=\Sigma_i\left(\frac{dc}{da_i}\frac{dZ}{db_i}-\frac{dc}{db_i}\frac{dZ}{da_i}\right) \quad . \quad . \quad . \quad . \quad (61.)$$

(see the last article). But, by Theorem V., this becomes

$$\frac{dc}{dt}=[c, Z]. \quad . \quad . \quad . \quad . \quad . \quad (62.)$$

It is worth observing that both this equation and (60.) might have been obtained indirectly as follows. Since c is constant, we have $c'=0$; that is, $\frac{dc}{dt}+[Z, c]=0$ (see (32.), art. 22.); this gives (62.), since $[Z, c]=-[c, Z]$, and again, by Theorem V., is changed into (61.); and if, in the latter, we put successively $c=a_j$, $c=b_j$, we obtain the system (60.).

SECTION V.—*On the Variation of Elements.*

52. The following general problem includes, I believe, all the cases which occur in practice. Let $P_1, \ldots P_n, Q_1, \ldots Q_n$ be any functions whatever of the $2n$ variables $x_1, \ldots x_n, y_1, \ldots y_n$ and t. It is required to express the $2n$ integrals of the system of $2n$ simultaneous differential equations of the first order

$$x'_i=P_i,\ y'_i=Q_i \quad . \quad . \quad . \quad . \quad . \quad (63.)$$

in the same form as the integrals (supposed given) of the canonical system

$$x'_i=\frac{dZ}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}, \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

by substituting functions of t for the constant elements of the latter system.

Suppose a *normal solution* (see end of art. 50.) of the system (I.) to be employed. The elements a_i, b_i represent the same functions of x_1, &c., y_1, &c. and t as before, but are now *variable*; consequently we have

$$a'_i=\frac{da_i}{dt}+\Sigma_j\left\{\frac{da_i}{dx_j}x'_j+\frac{da_i}{dy_j}y'_j\right\}=\frac{da_i}{dt}+\Sigma_j\left\{P_j\frac{da_i}{dx_j}+Q_j\frac{da_i}{dy_j}\right\},$$

with a similar expression for b'_i. But, by equations (57.) and (60.), these are immediately transformed into the following:

$$\left.\begin{aligned} a'_i&=\frac{dZ}{db_i}+\Sigma_j\left\{Q_j\frac{dx_j}{db_i}-P_j\frac{dy_j}{db_i}\right\} \\ b'_i&=-\frac{dZ}{da_i}-\Sigma_j\left\{Q_j\frac{dx_j}{da_i}-P_j\frac{dy_j}{da_i}\right\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (E.)$$

where Z, Q_j, P_j, x_j, y_j in the second members are supposed to be expressed (*Hyp.* I.) in terms of the elements and t. Thus the system (63.) is transformed into a system involving the new variables a_i, b_i, instead of the original variables x_i, y_i.

53. If, instead of employing a set of *normal integrals* of the pattern system (I.), we take *any* complete set of integrals $c_1, c_2, \ldots c_{2n}$, then c_1, &c. may be considered as

functions of a_1, &c., and again, through them, of the variables. We have then

$$c'_i=\frac{dc_i}{da_1}a'_1+\ldots+\frac{dc_i}{db_1}b'_1+\ldots;$$

and if in this equation the values of a'_1, &c. be introduced from the formula (E.) of the last article, the following expression results:

$$c'_i=\{Z, c_i\}+\Sigma_j(Q_j\{x_j, c_i\}-P_j\{y_j, c_i\})$$

(in which the symbol $\{p, q\}$ is used to denote

$$\Sigma_k\left\{\frac{dp}{db_k}\frac{dq}{da_k}-\frac{dp}{da_k}\frac{dq}{db_k}\right\},$$

so that by (59.) (Theorem V.) we have $\{p, q\}=-[p, q]$; but in $\{p, q\}$ p and q are considered as functions of a_1, &c., b_1, &c., whilst in $[p, q]$ they are considered as functions of x_1, &c., y_1, &c.). Now, considering p, q as functions of c_1, &c., and through these, of a_1, &c., we have (by reasoning exactly similar to that employed in deducing equation (24.), art. 9.)

$$\{p, q\}=\Sigma\left(\{c_\alpha, c_\beta\}\left(\frac{dp}{dc_\alpha}\frac{dq}{dc_\beta}-\frac{dp}{dc_\beta}\frac{dq}{dc_\alpha}\right)\right)$$

(the summation referring to all binary combinations of the indices α, β). Hence we have, putting $q=c_i$,

$$\{p, c_i\}=\Sigma_\alpha\left(\{c_\alpha, c_i\}\frac{dp}{dc_\alpha}\right), \quad \ldots \ldots \quad (64.)$$

and consequently the above expression for c'_i becomes

$$c'_i=\{Z, c_i\}+\Sigma_\alpha\Sigma_j\left(\{c_\alpha, c_i\}\left(Q_j\frac{dx_j}{dc_\alpha}-P_j\frac{dy_j}{dc_\alpha}\right)\right), \quad \ldots \ldots \quad (F.)$$

an equation which is easily seen to become identical with (E.), art. 52, when $c_1 \ldots c_{2n}$ represent $a_1 \ldots a_n$, $b_1 \ldots b_n$.

54. The simplest case is that in which the system of equations (63.), whose integrals are sought, are of the *canonical form*; that is, where

$$P_i=\frac{dW}{dy_i}, \quad Q_i=-\frac{dW}{dx_i},$$

W being a given function of the variables (with or without t). In this case the formula (E.) becomes

$$\left.\begin{aligned} a'_i&=\frac{dZ}{db_i}-\frac{dW}{db_i} \\ b'_i&=-\frac{dZ}{da_i}+\frac{dW}{da_i} \end{aligned}\right\} \quad \ldots \ldots \quad (65.)$$

whilst (F.) is easily found to be reducible, by the help of (64.), to either of the following forms:

$$c'_i=\{Z, c_i\}-\{W, c_i\} \quad \ldots \ldots \quad (66.)$$

$$c'_i=\Sigma_\alpha\left(\{c_\alpha, c_i\}\left(\frac{dZ}{dc_\alpha}-\frac{dW}{dc_\alpha}\right)\right). \quad \ldots \ldots \quad (67.)$$

If we put $W=Z+\Omega$, so that Ω may be called the "disturbing function," the above formulæ become

$$a'_i=-\frac{d\Omega}{db_i},\quad b'_i=\frac{d\Omega}{da_i}\quad \ldots\ldots\ldots \quad (68.)$$

$$c'_i=\Sigma_\alpha\left(\{c_i,\ c_\alpha\}\frac{d\Omega}{dc_\alpha}\right).\quad \ldots\ldots\ldots \quad (69.)$$

On the first of these forms see the note to art. 38. With respect to the form (69.), if we put for $\{c_i, c_\alpha\}$ its equivalent $-[c_i, c_\alpha]$, or $[c_\alpha, c_i]$ (see Theorem V. art. 49.), we obtain the well-known expression

$$c'_i=\Sigma_\alpha\left([c_\alpha,\ c_i]\frac{d\Omega}{dc_\alpha}\right).$$

The difference between this last form and (69.) consists in this; that in the latter the coefficients $[c_\alpha, c_i]$ are obtained from the expressions for c_1, c_2, &c. *in terms of the variables*; whereas in (69.) the coefficients $\{c_i, c_\alpha\}$ are similarly obtained from the expressions for c_1, &c. *in terms of the normal elements* a_1, &c., b_1, &c.*; and when a *normal solution* of the undisturbed problem has been obtained, the latter process will generally be found much more convenient than the former, since the elements c_1, &c. will usually be much simpler functions of the *normal elements* than of the *variables*.

55. In illustration of this, it will be worth while to deduce the expressions for the variations of the ordinary elliptic elements of a planet's orbit from those of the normal elements given in art. 30.

Let a and e be the semiaxis major and excentricity, ι the inclination of the orbit to a fixed ecliptic, ν the longitude of the node, ϖ the longitude of the perihelion, $nt+(\varepsilon)$ the mean longitude of the planet; *longitudes* being reckoned in the plane of the ecliptic (from a fixed origin) *as far as the node*, and then *on the plane of the orbit*. As usual, n stands for $\frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}$. Also let $nt+(\varepsilon)=\int_0^t ndt+\varepsilon$, so that $\varepsilon'=(\varepsilon)'+tn'$.

If, then, we call the six normal elements α_1, α_2, α_3, β_1, β_2, β_3, we have (see art. 30.)

$$\alpha_1=\frac{m\mu}{2a},\qquad \beta_1=\frac{\varpi-(\varepsilon)}{n},$$

$$\alpha_2=m\sqrt{\mu a(1-e^2)},\qquad \beta_2=\varpi-\nu,$$

$$\alpha_3=m\sqrt{\mu a(1-e^2)}.\cos\iota,\qquad \beta_3=\nu;$$

from which, conversely,

$$a=\frac{m\mu}{2\alpha_1},\qquad \varpi=\beta_2+\beta_3,$$

$$1-e^2=\frac{2\alpha_1\alpha_2^2}{m^3\mu^2},\qquad \nu=\beta_3,$$

$$\cos\iota=\frac{\alpha_3}{\alpha_2},\qquad (\varepsilon)=\beta_2+\beta_3-\frac{(2\alpha_1)^{\frac{3}{2}}}{m^{\frac{3}{2}}\mu}\beta_1.$$

* $\{c_i,\ c_\alpha\}=\Sigma_j\left(\frac{dc_i}{db_j}\frac{dc_\alpha}{da_j}-\frac{dc_i}{da_j}\frac{dc_\alpha}{db_j}\right).$

From these expressions the values of $\{a, e\}$, $\{a, \iota\}$, &c. are found with the greatest simplicity, and the results are

$$m\mu\{a, (\varepsilon)\}=2na^2, \quad m\mu\{(\varepsilon), e\}=\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2}),$$

$$m\mu\{\varpi, e\} =\frac{na\sqrt{1-e^2}}{e}, \quad m\mu\{(\varepsilon), \iota\}=\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2},$$

$$m\mu\{\varpi, \iota\} =\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2}, \quad m\mu\{\nu, \iota\}=\frac{na}{\sin\iota\sqrt{1-e^2}},$$

the rest all vanishing. Hence, observing that if R be taken in its usual signification we have $\Omega=-\mathrm{R}$, we obtain*

$$\mu a' =2na^2\frac{d\mathrm{R}}{d(\varepsilon)},$$

$$\mu e' =\frac{-na\sqrt{1-e^2}}{e}\left\{\frac{d\mathrm{R}}{d\varpi}+(1-\sqrt{1-e^2})\frac{d\mathrm{R}}{d(\varepsilon)}\right\},$$

$$\mu(\varepsilon)'=-2na^2\frac{d\mathrm{R}}{da}+\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{d\mathrm{R}}{de}+\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2}\frac{d\mathrm{R}}{d\iota},$$

$$\mu\varpi' =na\left\{\frac{\sqrt{1-e^2}}{e}\frac{d\mathrm{R}}{de}+\frac{1}{\sqrt{1-e^2}}\tan\frac{\iota}{2}\frac{d\mathrm{R}}{d\iota}\right\},$$

$$\mu\iota' =\frac{-na}{\sqrt{1-e^2}}\left\{\frac{1}{\sin\iota}\frac{d\mathrm{R}}{d\nu}+\tan\frac{\iota}{2}\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\},$$

$$\mu\nu' =\frac{na}{\sin\iota\sqrt{1-e^2}}\frac{d\mathrm{R}}{d\iota},$$

in which we may, as usual, put ε for (ε), provided that in forming the term $\frac{d\mathrm{R}}{da}$, nt be exempt from differentiation with respect to a.

56. A comparison of the above process with that by which the corresponding

* If we consider R as a function of p, q instead of ι, ν, where $p=\tan\iota\cos\nu$, $q=\tan\iota\sin\nu$, we find

$$\frac{d\mathrm{R}}{d\iota}=\sec^2\iota\left(\cos\nu\frac{d\mathrm{R}}{dp}+\sin\nu\frac{d\mathrm{R}}{dq}\right)$$

$$\frac{d\mathrm{R}}{d\nu}=\tan\iota\left(\cos\nu\frac{d\mathrm{R}}{dq}-\sin\nu\frac{d\mathrm{R}}{dp}\right),$$

and consequently

$$\mu p'=\frac{-na(\sec\iota)^2}{\sqrt{1-e^2}}\left\{\sec\iota\frac{d\mathrm{R}}{dq}+\tan\frac{\iota}{2}\cos\nu\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\}$$

$$\mu q'=\frac{na(\sec\iota)^2}{\sqrt{1-e^2}}\left\{\sec\iota\frac{d\mathrm{R}}{dp}-\tan\frac{\iota}{2}\sin\nu\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\}.$$

The formulæ will then agree with those of the Mécanique Céleste (Supplement to vol. iii. p. 360, ed. 1844), if we allow for the different mode of measuring longitudes, and neglect, as LAPLACE does, terms of the second order with respect to ι and $\frac{d\mathrm{R}}{d\iota}$. (LAPLACE uses R with the opposite sign.) Those in the text agree (allowing for notation) with the expressions given by Professor HANSEN, Astr. Nachr. No. 166, art. 3, equations (2).

expressions are obtained by PONTÉCOULANT*, will show the convenience of using the coefficients $\{c_i, c_j\}$ instead of $[c_i, c_j]$ (in PONTÉCOULANT's notation (c_i, c_j)).

[It will be observed that the formulæ for $(\varepsilon)'$, ϖ', ι' at the end of the last article, do not agree with those of PONTÉCOULANT (p. 330) for the variations of the corresponding quantities ε, ω, φ. The reason of this is as follows:—In PONTÉCOULANT's notation φ expresses the same as ι in this paper, and α the same as ν. But ω (the longitude of the perihelion) is not the same as ϖ; the former being measured *entirely in the plane of the orbit* from a radius vector, *fixed in that plane*†, and assumed as the origin of longitudes. Consequently ε, in PONTÉCOULANT (which we will call $\varepsilon_{\prime}$ for distinction), is not the same as (ε) in the present paper. In fact, if we equate the expressions for the mean anomaly in the two notations, we have

$$\varepsilon_{\prime}-\omega=(\varepsilon)-\varpi;$$

also it is evident that if we put β for the angle between the node and the origin from which ω is measured, we have $d\beta=-\cos\iota d\nu$, and $\varpi=\nu+\beta+\omega$, so that

$$d\varpi=d\omega+(1-\cos\iota)d\nu.$$

If then it were allowable to consider R as capable of being expressed as a function of ω and $\varepsilon_{\prime}$ instead of ϖ and (ε), and if we represented by (R) the expression for R so transformed, we should have

$$\frac{d\text{R}}{d\varpi}d\varpi+\frac{d\text{R}}{d(\varepsilon)}d(\varepsilon)+\&\text{c.}=\frac{d(\text{R})}{d\omega}d\omega+\frac{d(\text{R})}{d\varepsilon_{\prime}}d\varepsilon_{\prime}+\&\text{c.};$$

and if, in the two first terms, we put for $d\varpi$ and $d(\varepsilon)$ the values $d\varpi=d\omega+(1-\cos\iota)d\nu$, $d(\varepsilon)=d\varepsilon_{\prime}+(1-\cos\iota)d\nu$, and compare the two expressions, we find

$$\frac{d\text{R}}{d\varpi}=\frac{d(\text{R})}{d\omega},\quad \frac{d\text{R}}{d(\varepsilon)}=\frac{d(\text{R})}{d\varepsilon_{\prime}},$$

$$\frac{d\text{R}}{d\nu}+(1-\cos\iota)\left(\frac{d\text{R}}{d(\varepsilon)}+\frac{d\text{R}}{d\varpi}\right)=\frac{d(\text{R})}{d\nu}.$$

These relations, together with the equation

$$\varpi'=\omega'+(1-\cos\iota)\nu',$$

are easily seen to render the expressions at the end of art. 55 identical with those of PONTÉCOULANT; in fact, it is by an equivalent transformation that the latter are finally obtained by that author from the correct expressions in p. 328. But it is to be observed that this proceeding is founded upon a *false assumption*; for it is impossible to express R as a function of a, e, ι, ν, $\varepsilon_{\prime}$, ω, as is obvious from the consideration that R, in its original form, is not a function of $(\varepsilon)-\varpi$ merely, but also of (ε); whilst (ε) is *not expressible as a function of the new elements*, as is shown by the equation $d(\varepsilon)=d\varepsilon_{\prime}+(1-\cos\iota)d\nu$‡. It would be out of place to enter further into this sub-

* Théorie Anal. du Système du Monde, tome i. pp. 316–330.

† On the meaning of this expression, see below, art. 73.

‡ It would be a work of some trouble to trace *accurately* the process by which LAPLACE arrives at the for-

ject here, especially as some of the most important principles involved in it have been discussed elsewhere *. See also Appendix B.]

57. Returning to the expression (69.), art. 54, it may be observed that the coefficients $\{c_i, c_j\}$ are to be expressed in terms of c_1, c_2, &c., and this involves no difficulty when *each* of the two sets of elements c_1, &c., a_1, &c. can be expressed *in terms of the other explicitly*, as was the case in the example just discussed. Suppose, however, that the normal set a_1, &c., b_1, &c. are given in terms of the set c_1, &c., but that it is impracticable or inconvenient to obtain the converse equations expressing the latter in terms of the former. In this case we may proceed as follows.

Adopting the notation of art. 1 †, and putting f, g for any two of the set c_1, c_2, &c., we have

$$\{f, g\} = \Sigma_i \frac{d(f, g)}{d(b_i, a_i)};$$

suppose this equation written at length, and then, after multiplying by $\frac{d(b_j, a_j)}{d(f, g)}$, let each side be summed with respect to all binary combinations f, g. The result is (see art. 1, equation (4.)),

$$\Sigma\left(\frac{d(b_j, a_j)}{d(f, g)} \cdot \{f, g\}\right) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70.)$$

(the summation referring to the combinations f, g). Again, if the former equation be multiplied by $\frac{d(p, q)}{d(f, g)}$, where p, q represent any two of the normal elements, a_1, &c. b_1, &c., *except a conjugate pair*, and the sum be taken as before, we have

$$\Sigma\left(\frac{d(p, q)}{d(f, g)} \cdot \{f, g\}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71.)$$

The two formulæ (70.), (71.) give $n(2n-1)$ linear equations for determining the $n(2n-1)$ unknown quantities $\{f, g\}$; the coefficients of the latter being all given functions of c_1, &c. But such cases will hardly occur in practice. (With respect to the form of the above system of linear equations, it is easy to show that the complete determinant of the coefficients is $=1$.)

58. The integration of the formulæ (65.), art. 53, would give the means of expressing the solution of the system

$$x_i' = \frac{dW}{dy_i}, \quad y_i' = -\frac{dW}{dx_i}$$

mulæ alluded to in a preceding note, as the various steps of it are to be found in different places, the notation is somewhat inconsistent, and the results *do not profess to be rigorous*. My impression is, however, that Laplace nowhere commits the fallacy of assuming (for example) that R is a function of r, v, z, or r, v, s (see vol. i. p. 295), where v is the angle described by the radius vector on the varying plane of the orbit.

* See Jacobi's two letters to Professor Hansen in Crelle's Journal, vol. xlii.

† *i. e.* using $\frac{d(u, v)}{d(x, y)}$ as an abbreviation for $\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$.

in the *form* of a normal solution of any other similar system

$$x_i'=\frac{dZ}{dy_i},\quad y_i'=-\frac{dZ}{dx_i},$$

which may be chosen as the pattern.

In the most usual examples the function to be chosen for Z is naturally suggested by the circumstance, that W presents itself under the form of the sum of two functions $Z+\Omega$, of which the former, taken alone, gives an integrable system. But this is not necessarily the case; and it is worth while to observe that the formulæ (65.) take a simple and remarkable form whatever Z may be, provided that it be a function *not containing* t *explicitly*. For then, assuming the "integral of vis viva," $Z=h$, as one of the normal integrals of the pattern system*, the element conjugate to h is τ (the constant added to t); and observing that Z, in (65.), being *expressed in terms of the elements*, reduces itself simply to h, we shall have $\frac{dZ}{dh}=1$, whilst the differential coefficients of Z with respect to all the other elements vanish; so that, if we put $a_1, \dots a_{n-1}, b_1, \dots b_{n-1}$ for the remaining elements, the system (65.) takes the following form:—

$$\left.\begin{aligned} h'&=-\frac{dW}{d\tau}, & \tau'&=-1+\frac{dW}{dh} \\ a_i'&=-\frac{dW}{db_i}, & b_i'&=\frac{dW}{da_i} \end{aligned}\right\} \quad \dots\dots\dots \quad (72.)$$

This, in dynamics, gives the process to be used in the following problem: "*To express the solution of any dynamical problem in the form of the solution of any other* (*involving the same number of variables*) *in which the principle of vis viva subsists.*"

59. As an example of the above process we may apply it to determine the motion of a simple free pendulum (not taking into account the earth's rotation).

Let l be the length of the pendulum, and let the mass of the material point m placed at its extremity be represented by unity. Also let x, y, z be the rectangular coordinates of m, the origin being at the position of rest of m, and the axis of z directed vertically upwards. The equation to the sphere described by m is

$$x^2+y^2+z^2-2lz=0,$$

and the *force-function* U is $-gz$.

Hence if we take, as the two independent coordinates, the radius vector ϱ of the projection of m on the plane of xy, and the angle θ between ϱ and the axis of x, we shall have for the differential equations of motion,

$$\left.\begin{aligned} \varrho'&=\frac{dW}{du}, & \theta'&=\frac{dW}{dv} \\ u'&=-\frac{dW}{d\varrho}, & v'&=-\frac{dW}{d\theta} \end{aligned}\right\} \quad \dots\dots\dots \quad (A.)$$

* See art. 19 (where h_i in equation (29.) is a misprint for b_i).

where $u, v,$ are the variables conjugate respectively to ϱ, θ, and defined by the equations

$$u=\frac{d\mathrm{T}}{d\varrho'},\quad v=\frac{d\mathrm{T}}{d\theta'};$$

and W is T—U expressed in terms of ϱ, θ, u, v.

Now $x=\varrho\cos\theta$, $y=\varrho\sin\theta$, $z=l-\sqrt{l^2-\varrho^2}$; hence

$$\mathrm{T}\left(=\frac{1}{2}(x'^2+y'^2+z'^2)\right)=\frac{1}{2}\left\{\frac{l^2}{l^2-\varrho^2}\varrho'^2+\varrho^2\theta'^2\right\},$$

from which the following expression for W is easily obtained:

$$\mathrm{W}=\frac{1}{2}\left(\frac{l^2-\varrho^2}{l^2}u^2+\frac{v^2}{\varrho^2}\right)+g(l-\sqrt{l^2-\varrho^2}). \quad \ldots\ldots \quad (\mathrm{W.})$$

Now let us take as a model for the solution of the above system, a set of normal integrals (in polar coordinates) of the system

$$x''+n^2x=0,\quad y''+n^2y=0, \quad \ldots\ldots \quad (\mathrm{B.})$$

where $n^2=\frac{g}{l}$. In this system we have $\mathrm{U}=-\frac{1}{2}n^2\varrho^2$; and proceeding exactly as in art. 27, we obtain the following results: the two integrals of *vis viva* and of areas are

$$\left.\begin{aligned} h&=\frac{1}{2}\left(u^2+\frac{v^2}{\varrho^2}+n^2\varrho^2\right)\\ c&=v \end{aligned}\right\} \quad \ldots\ldots \quad (\mathrm{i.})$$

these are to be solved for u, v; and then V is to be obtained from the equation $d\mathrm{V}=ud\varrho+vd\theta$. This gives

$$\mathrm{V}=c\theta+\int d\varrho\left\{2h-n^2\varrho^2-\frac{c^2}{\varrho^2}\right\}^{\frac{1}{2}};$$

and the remaining integrals are given by the equations

$$\frac{d\mathrm{V}}{dh}=t+\tau,\quad \frac{d\mathrm{V}}{dc}=\varpi,$$

τ and ϖ being the elements conjugate respectively to h and c. Performing the differentiations *first*, and taking the integrals in the second term so as to vanish with the expression $\left\{2h-n^2\varrho^2-\frac{c^2}{\varrho^2}\right\}^{\frac{1}{2}}$ (see Appendix A.), we find easily the final equations

$$\left.\begin{aligned} n^2\varrho^2&=h+\sqrt{h^2-n^2c^2}.\cos 2n(t+\tau)\\ c^2\varrho^{-2}&=h-\sqrt{h^2-n^2c^2}.\cos 2(\theta-\varpi) \end{aligned}\right\}, \quad \ldots\ldots \quad (\mathrm{ii.})$$

in which ϖ is the angle between the axis of x and a (distant) apse, and $-\tau$ is the time of passage through that apse. The four equations (i.), (ii.) comprise a complete normal solution of the equations (B.). The last is the polar equation to the elliptic orbit; and if we call a, b the semiaxes of the ellipse, we have

$$c=nab,\quad h=n^2\frac{a^2+b^2}{2}.$$

60. The solution of the system (A.) of the last article will now be expressed by the same equations (i.), (ii.), if the elements h, c, τ, ϖ be variables defined by the system (see art. 58.)

$$h'=-\frac{dW}{d\tau},\quad \tau'=-1+\frac{dW}{dh}$$

$$c'=-\frac{dW}{d\varpi},\quad \varpi'=\frac{dW}{dc},$$

where W is to be obtained by substituting in the expression (W.), art. 59, the values of ϱ, θ, u, v in terms of the elements and t, derived from equations (i.), (ii.). The result of this substitution is

$$W=\frac{h}{2}+\frac{c^2}{4l^2}-\frac{h^2}{4n^2l^2}-\frac{1}{2}\sqrt{h^2-n^2c^2}.\cos 2n(t+\tau)$$

$$+\frac{h^2-n^2c^2}{4n^2l^2}\cos 4n(t+\tau)+n^2l(l-\sqrt{l^2-\varrho^2}),$$

in which the value of ϱ^2 in the last term must be understood to be substituted from the first of equations (ii.). If we call ϕ the angle between the pendulum and the vertical, we shall have evidently

$$n^2l(l-\sqrt{l^2-\varrho^2})=n^2l^2(1-\cos\phi),$$

and the differential coefficient of this term with respect to any constant k involved in the value of ϱ will be $\frac{n^2}{2\cos\phi}\cdot\frac{d(\varrho^2)}{dk}$. Observing this, we obtain the following expressions for the variations of the elements:

$$h'=-n(\sec\phi-1)\sqrt{h^2-n^2c^2}.\sin 2n(t+\tau)+\frac{h^2-n^2c^2}{nl^2}\sin 4n(t+\tau)$$

$$\tau'=-\frac{h}{2n^2l^2}+\frac{1}{2}(\sec\phi-1)\left(1+\frac{h}{\sqrt{h^2-n^2c^2}}\cos 2n(t+\tau)\right)+\frac{h}{2n^2l^2}\cos 4n(t+\tau)$$

$$c'=0$$

$$\varpi'=\frac{c}{2l^2}-\frac{1}{2}(\sec\phi-1)\frac{n^2c}{\sqrt{h^2-n^2c^2}}\cos 2n(t+\tau)-\frac{c}{2l^2}\cos 4n(t+\tau).$$

The third of these equations gives ab=*constant*; hence, by means of the equations at the end of art. 59, the following expressions are easily deduced:

$$\frac{a'}{a}=-\frac{b'}{b}=\frac{n}{2}\left\{-(\sec\phi-1)\sin 2n(t+\tau)+\frac{a^2-b^2}{l^2}\sin 4n(t+\tau)\right\}$$

$$\tau'=\frac{a^2+b^2}{4l^2}\left(-1+\cos 4n(t+\tau)\right)+\frac{1}{2}(\sec\phi-1)\left(1+\frac{a^2+b^2}{a^2-b^2}\cos 2n(t+\tau)\right)$$

$$\varpi'=nab\left\{\frac{1-\cos 4n(t+\tau)}{2l^2}-(\sec\phi-1)\frac{\cos 2n(t+\tau)}{a^2-b^2}\right\}.$$

These equations are rigorous, and in general not easier to integrate than the original system of which they are a transformation; but they may be integrated approximately on particular hypotheses. For instance, if the pendulum never deviates much from the vertical, $\frac{\varrho^2}{l^2}$ is always small, and $\sec\phi-1=\frac{\varrho^2}{2l^2}$ nearly; introducing this value, and substituting for ϱ^2, we have for the *non-periodic* parts of the above expressions

$$\frac{a'}{a}=-\frac{b'}{b}=0$$

$$\tau'=-\frac{a^2+b^2}{16l^2}, \quad \varpi'=\frac{3}{8}\frac{nab}{l^2}.$$

Hence the axes of the mean ellipse are constant; also if T be the time of describing the ellipse, we shall have approximately $T=\frac{2\pi}{n}\cdot\frac{1}{1+\tau'}$, whence it follows that the motion of the apse during this period will be $\frac{3nab}{8l^2}\cdot\frac{2\pi}{n(1+\tau')}$, or $\frac{3ab}{8l^2}\cdot 2\pi$ nearly.

This agrees with the statement of the Astronomer Royal*. The above approximations would cease to be sufficiently accurate if b were very nearly equal to a, or the motion very nearly circular; but they apply to any other case, and in particular to that in which b is very small, or the motion nearly rectilinear. In the case of nearly circular motion, it would be necessary to develope $\sec\phi$ in a series of powers of a^2-b^2, and the results would be applicable whether ϕ were small or not. But I shall not pursue this subject further here.

SECTION VI.—*On the Transformation of Variables.*

61. The method of the variation of elements, theoretically considered, consists merely in a transformation of variables of a particular kind; that kind namely, which leads to a new system of differential equations belonging to the *same general class* as the original system. But practically, the choice of variables is determined by the well-known considerations from which the method derives its name.

It is the object of the present section to consider the general class of transformations of which the method in question is a particular, and not the only useful case.

62. *Definition of Normal Transformations.*

Let $\xi_1, \xi_2, \ldots \xi_n, \eta_1, \eta_2, \ldots \eta_n$ be new variables connected with the original variables x_1, &c. y_1, &c. by $2n$ equations (which may also involve t explicitly), such that each variable of either set may be considered as a function of the variables of the other set (with or without t). Let P be any function of $\xi_1, \xi_2, \ldots \xi_n, y_1, y_2, \ldots y_n$ and t; then if the equations connecting the old and new variables can be put in the form

$$\frac{dP}{dy_i}=x_i, \quad \frac{dP}{d\xi_i}=\eta_i, \qquad \ldots \ldots \ldots \quad (73.)$$

I propose to call the transformation *normal.*

* Proceedings of the Royal Astronomical Society, vol. xi. p. 160.

[It follows from Theorem I. art. 49, that the system (73.) is equivalent to each of the following:

$$\frac{dQ}{dx_i}=y_i, \quad \frac{dQ}{d\xi_i}=-\eta_i,$$

in which $Q=-P+\Sigma_i(x_i y_i)$, and is expressed in terms of $x_1, \ldots x_n, \xi_1, \ldots \xi_n$; or

$$\frac{dR}{d\eta_i}=\xi_i, \quad \frac{dR}{dy_i}=-x_i,$$

in which $R=-P+\Sigma_i(\xi_i\eta_i)$, and is expressed in terms of $y_1, \ldots y_n, \eta_1, \ldots \eta_n$; or lastly,

$$\frac{dS}{d\eta_i}=\xi_i, \quad \frac{dS}{dx_i}=y_i,$$

in which $S=-P+\Sigma_i(x_i y_i+\xi_i\eta_i)$, and is expressed in terms of $x_1, \ldots x_n, \eta_1, \ldots \eta_n$.

Any one of these forms might be used; but I shall employ the form (73.) for reasons of convenience.]

63. Inasmuch as the equations (73.) of the last article are of the same general form as the system (54.), art. 49, all the conclusions deduced from that form will subsist, *mutatis mutandis*; so that we may apply the Theorems (II.), (III.), (IV.), (V.), art. 49, by merely changing X into P, and

$x_1, \ldots x_n, y_1, \ldots y_n, a_1, \ldots a_n, b_1, \ldots b_n$, respectively into
$\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n, y_1, \ldots y_n, x_1, \ldots x_n$;

observing that instead of x'_i, y'_i we must now write $\frac{d\xi_i}{dt}$, $\frac{d\eta_i}{dt}$*. We thus obtain the following relations:

$$\frac{d\xi_i}{dt}=\frac{d\Psi}{d\eta_i}, \quad \frac{d\eta_i}{dt}=-\frac{d\Psi}{d\xi_i}, \quad \ldots \ldots \ldots \quad (74.)$$

where Ψ is a function of ξ_1, &c., η_1, &c. and t, defined by the equation

$$\Psi=-\left(\frac{dP}{dt}\right), \quad \ldots \ldots \ldots \quad (75.)$$

the brackets indicating that the expressions for $y_1, \ldots y_n$ in terms of the new variables ξ_1, &c., η_1, &c., are to be substituted in $\frac{dP}{dt}$ after the differentiation with respect to t; which is performed so far as t appears explicitly in the original expression for P as a function of $\xi_1 \ldots \xi_n, y_1 \ldots y_n$ and t. (See Theorem II.)

We have also the system

$$\left.\begin{aligned} \frac{d\xi_i}{dy_j}&=-\frac{dx_j}{d\eta_i}, \quad \frac{d\xi_i}{dx_j}=\frac{dy_j}{d\eta_i} \\ \frac{d\eta_i}{dy_j}&=\frac{dx_j}{d\xi_i}, \quad \frac{d\eta_i}{dx_j}=-\frac{dy_j}{d\xi_i} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (76.)$$

* For in the original theorems x'_i is the same thing as the differential coefficient of x_i taken with respect to t, as t appears explicitly in the expression for x_i in terms of a_1, &c., b_1, &c. and t; the analogous quantity in the present case is therefore the differential coefficient of ξ_i taken with respect to t, as t appears explicitly in the expression for ξ_i in terms of x_1, &c., y_1, &c. and t. But this must not now be denoted by ξ'_i, inasmuch as x_1, &c., y_1, &c. are themselves afterwards to be considered as functions of t.

(see Theorem III.). And if p, q be any two functions of the variables (with or without t), then

$$\Sigma_i\left(\frac{dp}{dy_i}\frac{dq}{dx_i}-\frac{dp}{dx_i}\frac{dq}{dy_i}\right)=\Sigma_i\left(\frac{dp}{d\eta_i}\frac{dq}{d\xi_i}-\frac{dp}{d\xi_i}\frac{dq}{d\eta_i}\right), \quad \ldots\ldots \quad (77.)$$

where p and q in the first member are supposed to be expressed in terms of x_1, &c., y_1, &c., and, in the second, in terms of ξ_1, &c., η_1, &c. In other words, the value of $[p, q]$ is the same, whether it be obtained from the expressions for p, q in terms of the original variables, or by an analogous process from their expressions in terms of the new.

Particular cases of (77.) are the relations

$$[\xi_i, \eta_i]=-1, \quad [\xi_i, \xi_j]=[\eta_i, \eta_j]=[\xi_i, \xi_j]=0. \quad \ldots\ldots \quad (78.)$$

(See Theorems IV., V.)

64. The relations (74.), (76.), (77.), (78.) of the last article depend solely upon the form of the equations (73.), art. 62, which connect the new variables with the old; and are independent of any supposition as to the equations which may determine either set of variables as functions of t. Let us now, however, introduce the supposition that the original variables $x_1, \ldots x_n, y_1, \ldots y_n$ are determined as functions of t by the system of differential equations,

$$x'_i=\frac{dZ}{dy_i}, \quad y'_i=-\frac{dZ}{dx_i}. \quad \ldots\ldots \quad (\text{I.})$$

The relations just established enable us immediately to transform this system into another involving the new variables instead of the old; for we have

$$\xi'_i=\frac{d\xi_i}{dt}+\Sigma_j\left(\frac{d\xi_i}{dx_j}x'_j+\frac{d\xi_i}{dy_j}y'_j\right);$$

now

$$\frac{d\xi_i}{dt}=\frac{d\Psi}{d\eta_i} \text{ (see (74.), art. 63);}$$

and if in the remaining term we substitute for x'_j, y'_j their values from (I.), and for $\frac{d\xi_i}{dx_j}$, $\frac{d\xi_i}{dy_j}$ their values $\frac{dy_j}{d\eta_i}$, $-\frac{dx_j}{d\eta_i}$, it becomes

$$\Sigma_j\left(\frac{dZ}{dx_j}\frac{dx_j}{d\eta_i}+\frac{dZ}{dy_j}\frac{dy_j}{d\eta_i}\right),$$

which is equivalent to $\frac{dZ}{d\eta_i}$, if Z be supposed expressed in terms of the new variables.

We have then

$$\xi'_i=\frac{d\Psi}{d\eta_i}+\frac{dZ}{d\eta_i},$$

and, exactly in the same way,

$$\eta'_i=-\frac{d\Psi}{d\xi_i}-\frac{dZ}{d\xi_i}.$$

This result may be stated in the form of the following *Theorem* VIII*. If the system

* This theorem, in its general form, is, to the best of my knowledge, new. But that case of it in which P does not contain t explicitly has already been proved in a different way by M. DESBOVES, who has, by means

of differential equations (I.) be transformed by the introduction of new variables $\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n$, connected with the original variables $x_1, \ldots x_n, y_1, \ldots y_n$, by the equations $\frac{dP}{dy_i}=x_i$, $\frac{dP}{d\xi_i}=\eta_i$, where P is any function of $\xi_1, \ldots \xi_n, y_1, \ldots y_n$, which may also contain t explicitly, then the transformed equations are

$$\xi_i'=\frac{d\Phi}{d\eta_i}, \quad \eta_i'=-\frac{d\Phi}{d\xi_i}, \quad \ldots \ldots \ldots \quad (79.)$$

in which Φ is defined by the equation

$$\Phi=Z-\frac{dP}{dt},$$

and is to be expressed in terms of the new variables. (The substitution of the new variables in $\frac{dP}{dt}$ is to be made *after* the differentiation. See art. 63.).

Corollary.—If P do not contain t explicitly, $\frac{dP}{dt}=0$ and $\Phi=Z$; so that in this case the transformation is effected merely by expressing Z in terms of the new variables.

65. It follows from (77.), art 63, that if f, g be any two integrals of the system (I.), the value of $[f, g]$ is the same whether it be derived from these integrals in their original form, or similarly obtained from the same integrals after transformation by the introduction of the new variables. And consequently *if* n *integrals* $a_1, a_2, \ldots a_n$ *of the original system be given, which satisfy the* $\frac{n(n-1)}{2}$ *conditions* $[a_i, a_j]=0$, *they will continue, after a normal transformation, to satisfy the analogous conditions,* so that the method of finding the remaining integrals given in Theorem VII. art. 49, will also continue to be applicable. We had an instance of this in the case of the problem of central forces (art. 27.), where the above conditions were found to subsist after the transformation from rectangular to polar coordinates. (It will be shown presently that every transformation of coordinates is a *normal transformation.*)

66. It was shown in Part I. (art. 18.), that if W be any function of $x_1, x_2, \ldots x_n$, $x_1', x_2', \ldots x_n'$ (which may also contain t explicitly), the system of n differential equations of the second order

$$\left(\frac{dW}{dx_i'}\right)'=\frac{dW}{dx_i} \quad \ldots \ldots \ldots \quad (80.)$$

may be changed into a system of $2n$ equations of the first order of the form (I.),

of it, deduced JACOBI's form of the method of the Variation of Elements (namely, the equations (68.), art. 54), from the similar form of LAGRANGE, in which the elements are the initial values of the variables. It will appear in the sequel that the extension to the case in which P may contain t, is of importance. If the expression were not already appropriated, I should have proposed definitively to call P the "modulus of transformation;" and I shall use this term provisionally in the present paper, not being able to suggest a tolerable substitute. After all, as the word "modulus" itself is used without confusion in very different senses according to the subject matter, there is, perhaps, no reason why a similar liberty should not be allowed in the use of the proposed expression.

art. 64, by putting $y_i=\frac{dW}{dx'_i}$, and taking Z a function of x_1, &c., y_1, &c. and t, defined by the equation

$$Z=-W+\Sigma_i(x'_i y_i),$$

in which $x'_1, \ldots x'_n$ are to be expressed in terms of y_1, &c., x_1, &c.*.

Conversely, a system of the form (I.) being given, it may be changed into a system of the form (80.) as follows: by means of the equations $x'_i=\frac{dZ}{dy_i}$, let $y_1, \ldots y_n$ be expressed in terms of x'_1, &c., x_1, &c.; it follows from Theorem I. (art. 49.) that we shall have

$$y_i=\frac{dW}{dx'_i} \quad \ldots \ldots \quad \text{(a.)}$$

and

$$\frac{dZ}{dx_i}=-\frac{dW}{dx_i}, \quad \ldots \ldots \quad \text{(b.)}$$

where W is a function of x_1, &c., x'_1, &c. defined by equation

$$W=-Z+\Sigma_i(x'_i y_i),$$

in which $y_1, \ldots y_n$ are to be expressed in terms of x'_1, &c., x_1, &c. The n equations $y'_i=-\frac{dZ}{dx_i}$ are then changed by (a.) and (b.) into the form (80.).

On the Transformation of Coordinates.

67. It has been seen in the preceding article, that we can always change the system of $2n$ equations of the first order of the form (I.), art. 64, into a system of n equations of the second order of the form (80.). In this latter form the equations of dynamics naturally present themselves.

Now in the case of the dynamical equations, $x_1, x_2, \ldots x_n$ are the independent *coordinates* of the system (the word *coordinates* being taken in its most general sense), and when the equations are to be changed into the form (I.), the additional variables $y_1, \ldots y_n$ are defined by the equations $\frac{dW}{dx'_i}=y_i$. In this case a *transformation of coordinates*, in the most general sense, consists in taking n new variables $\xi_1, \xi_2, \ldots \xi_n$, connected with the original coordinates $x_1, \ldots x_n$ by n equations, which may also involve t explicitly. It is a well-known theorem, that the transformation of the equations (80.) is effected merely by expressing W in terms of the new coordinates $\xi_1, \ldots \xi_n$, and their differential coefficients $\xi'_1, \ldots \xi'_n$, instead of the old; so that the new equations are

$$\left(\frac{dW}{d\xi'_i}\right)'=\frac{dW}{d\xi_i}: \quad \ldots \ldots \quad \text{(81.)}$$

the proof of this theorem does not depend upon the form of the function W; and we know also (see arts. 18 & 66.), that whatever be the form of W, these new equations may be again transformed to a system of the form (I.), by taking n additional

* This theorem is a generalization of Sir W. R. HAMILTON's transformation of the Dynamical Equations. See Part I. art. 18.

variables $\eta_1, \ldots \eta_n$, defined by the equations

$$\eta_i = \frac{d\mathrm{W}}{d\xi'_i}.$$

[It is to be observed that this last transformation is not, in general, equivalent to expressing the original Z in terms of the new variables; for the original Z is $-\mathrm{W}+\Sigma(x'_i y_i)$ (art. 66.), and the analogous expression derived from (81.) is $-\mathrm{W}+\Sigma(\xi'_i \eta_i)$, which is not, in general, equivalent to the former. It will be seen presently that the two expressions *are* equivalent when the equations connecting $x_1, \ldots x_n$ with $\xi_1, \ldots \xi_n$ do not involve t explicitly.]

68. Following the analogy of the dynamical equations, I shall adopt the following as the

Definition of a transformation of coordinates.

The original equations (I.), art. 64, having been changed into the form (80.), art. 66, let $\xi_1, \xi_2, \ldots \xi_n$ be n new variables connected with the n variables $x_1, x_2, \ldots x_n$ by n equations, which may also involve t explicitly; and let $\eta_1, \eta_2, \ldots \eta_n$ be n other new variables defined by the equations $\frac{d\mathrm{W}}{d\xi'_i} = \eta_i$ (where W has been expressed in terms of ξ_1, &c., ξ'_1, &c.). By means of the $2n$ assumed relations, the $2n$ original variables $x_1, \ldots x_n, y_1, \ldots y_n$ can be expressed as functions of the $2n$ new variables $\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n$. Let this substitution be called a "*transformation of coordinates.*"

It has been seen in the preceding article that the original equations (I.) are changed by a transformation of coordinates into a system of the *same form*, which however cannot in general be obtained by merely expressing Z in terms of the new variables. But we are not at liberty to assume (and it is not generally true) that a change of the system (I.) into another of the same form is a *normal transformation* (art. 62.). It has already been stated, however (end of art. 65.), that this *is true in the present case*; a proposition which I proceed to establish.

69. *Every transformation of coordinates is a normal transformation.*

To prove this theorem, we have to show that every transformation of the kind described in the last article is also of the kind defined in art. 62; in other words, that it is possible to assign a function P, of $\xi_1, \ldots \xi_n, y_1, \ldots y_n$ (with or without t), such that the given relations between x_1, &c., ξ_1, &c., which define the transformation of coordinates, shall be equivalent to the system of equations

$$\frac{d\mathrm{P}}{dy_i} = x_i, \quad \frac{d\mathrm{P}}{d\xi_i} = \eta_i.$$

Take $$\mathrm{P} = (x_1)y_1 + (x_2)y_2 + \ldots + (x_n)y_n \quad \ldots \ldots \quad (82.)$$

(the brackets indicating that $x_1, \ldots x_n$ are to be expressed in terms of $\xi_1, \ldots \xi_n$; so that P is a function of $\xi_1, \ldots \xi_n, y_1, \ldots y_n$, with or without t according as the equations connecting $x_1, \ldots x_n$ with $\xi_1, \ldots \xi_n$ do or do not contain t explicitly). Then P is the function required. For we have, at once, $\frac{dP}{dy_i}=x_i$; also

$$\frac{dP}{d\xi_i}=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\ldots+y_n\frac{d(x_n)}{d\xi_i};$$

now the definition of η_i is $\eta_i=\frac{dW}{d\xi'_i}$, where

$$W=-(Z)+x'_1y_1+x'_2y_2+\ldots+x'_ny_n$$

(expressed in terms of $\xi_1,\ldots\xi_n$, $\xi'_1,\ldots\xi'_n$ (art. 66.)); hence, putting Z (without brackets) for the original form of Z in terms of x_1, &c., y_1, &c., and observing that (Z) becomes a function of ξ'_1, &c., only through y_1, &c., we have

$$\begin{aligned}\frac{dW}{d\xi'_i}=&-\frac{dZ}{dy_1}\frac{dy_1}{d\xi'_i}-\frac{dZ}{dy_2}\frac{dy_2}{d\xi'_i}-\ldots-\frac{dZ}{dy_n}\frac{dy_n}{d\xi'_i}\\&+x'_1\frac{dy_1}{d\xi'_i}+x'_2\frac{dy_2}{d\xi'_i}+\ldots+x'_n\frac{dy_n}{d\xi'_i}\\&+y_1\frac{dx'_1}{d\xi'_i}+y_2\frac{dx'_2}{d\xi'_i}+\ldots+y_n\frac{dx'_n}{d\xi'_i}.\end{aligned}$$

The two first lines of this expression vanish by virtue of the equations $x'_i=\frac{dZ}{dy_i}$; and since

$$x'_j=\frac{d(x_j)}{dt}+\frac{d(x_j)}{d\xi_1}\xi'_1+\frac{d(x_j)}{d\xi_2}\xi'_2+\ldots+\frac{d(x_j)}{d\xi_n}\xi'_n,$$

we have

$$\frac{dx'_j}{d\xi'_i}=\frac{d(x_j)}{d\xi_i};$$

so that we have, finally,

$$\eta_i=\frac{dW}{d\xi'_i}=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\ldots+y_n\frac{d(x_n)}{d\xi_i},$$

which is evidently equivalent (see the expression (82.)) to

$$\eta_i=\frac{dP}{d\xi_i}:$$

the proposition in question is thus established, and may be enunciated as follows :—

70. *Theorem* IX.—*Every transformation of coordinates is a normal transformation, of which the modulus* * P *is a function of* $\xi_1,\ldots\xi_n$, $y_1,\ldots y_n$ (*with or without* t) *given by the equation*

$$P=(x_1)y_1+(x_2)y_2+\ldots+(x_n)y_n$$

(*the brackets indicating that* $x_1,\ldots x_n$ *are to be expressed in terms of* $\xi_1,\ldots\xi_n$). (See arts. 62, 68.)

This theorem will be made more intelligible by applying it to a very simple example.

Let it be proposed, then, to transform from rectangular to polar coordinates the differential equations of any dynamical problem referring to the motion of a single material point whose mass is *m*. Let x, y, z be the rectangular coordinates of *m*,

* See note on Theorem VIII. art. 64.

and u, v, w the variables conjugate to them; so that, putting $\mathrm{T}=\frac{1}{2}m(\mathrm{x}'^2+\mathrm{y}'^2+\mathrm{z}'^2)$, we have $\mathrm{u}=\frac{d\mathrm{T}}{d\mathrm{x}'}$, &c.; whence $\mathrm{T}=\frac{1}{2m}(\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2)$, and the equations of motion are $\mathrm{x}'=\frac{d\mathrm{Z}}{d\mathrm{u}}$, $\mathrm{u}'=-\frac{d\mathrm{Z}}{d\mathrm{x}}$, &c., where $\mathrm{Z}=\frac{1}{2m}(\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2)-\mathrm{U}$, and U is a given function of x, y, z, with or without t.

Now let r, θ, ϕ be polar coordinates of P, so that

$$\mathrm{x}=r\sin\theta\cos\phi,\quad \mathrm{y}=r\sin\theta\sin\phi,\quad \mathrm{z}=r\cos\theta.$$

Let u, v, w be the variables conjugate to r, ϑ, ϕ. Then the ordinary process of transformation would be as follows:—

(1) to express x′, y′, z′ in terms of $r, \theta, \phi, r', \theta', \phi'$, and thus transform T into a function of the latter quantities;

(2) to define u, v, w by the equations

$$u=\frac{d\mathrm{T}}{dr'},\quad v=\frac{d\mathrm{T}}{d\theta'},\quad w=\frac{d\mathrm{T}}{d\phi'},$$

and by means of these relations to express r', θ', ϕ' in terms of u, v, w, r, θ, ϕ, so that x′, y′, z′, and therefore, finally, T and Z, might be expressed as functions of the six new variables.

Instead of this, let us adopt the method indicated by the theorem at the beginning of this article.

We have then, for the modulus of transformation,

$$\mathrm{P}=(\mathrm{x})\mathrm{u}+(\mathrm{y})\mathrm{v}+(\mathrm{z})\mathrm{w},$$

in which x, y, z are to be expressed in terms of r, θ, ϕ; so that the proper form of P is

$$\mathrm{P}=\mathrm{u}r\sin\theta\cos\phi+\mathrm{v}r\sin\theta\sin\phi+\mathrm{w}r\cos\theta,$$

and the equations (corresponding to $\eta_i=\frac{d\mathrm{P}}{d\xi_i}$ (art. 69.)) which define the new variables u, v, w, are

$$u=\frac{d\mathrm{P}}{dr},\quad v=\frac{d\mathrm{P}}{d\theta},\quad w=\frac{d\mathrm{P}}{d\phi}.$$

These give

$$u=\sin\theta(\mathrm{u}\cos\phi+\mathrm{v}\sin\phi)+\mathrm{w}\cos\theta,$$

$$v=r\cos\theta(\mathrm{u}\cos\phi+\mathrm{v}\sin\phi)-r\mathrm{w}\sin\theta,$$

$$w=-r\sin\theta(\mathrm{u}\sin\phi-\mathrm{v}\cos\phi),$$

from which the values of u, v, w are easily obtained in terms of the six new variables. But in order to effect the transformation of T, we have only to square each side of these equations, and add them, after dividing the second by r^2 and the third by $(r\sin\theta)^2$; we thus obtain

$$\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2=u^2+\frac{v^2}{r^2}+\frac{w^2}{(r\sin\theta)^2},$$

and the transformed equations of motion are therefore

$$mr'=u,\ m\theta'=vr^{-2},\ m\varphi'=w(r\sin\theta)^{-2},$$

$$mu'=r^{-3}(v^2+w^2(\sin\theta)^{-2})+m\frac{dU}{dr},$$

$$mv'=(r\sin\theta)^{-3}w^2\cos\theta+m\frac{dU}{d\theta},\ \ w'=\frac{dU}{d\varphi}.$$

The preceding process is not, of course, given for the sake of the result, which may easily be verified directly, but in order to illustrate the meaning of the theorem on which it depends. It is hardly necessary to add, that if the problem involved the independent rectangular coordinates of any number of material points, the transformation to polar coordinates would be effected in the same way, by merely adding to P analogous terms for each point.

71. Before proceeding further there is an important remark to be made.

It has been hitherto assumed that the modulus of transformation P was a function of no other quantities than the $2n$ variables $\xi_1,\dots\xi_n, y_1,\dots y_n$ and t. But if every step of the demonstrations of the theorems of transformation which have been given (art. 62, &c.) be examined, it will be seen that they continue to hold good in the following more general form.

Take $$P=f(\xi_1,\xi_2,\dots\xi_n,y_1,y_2,\dots y_n,p,q,r,\dots,t),$$

where $p, q, r,\dots$ are any functions of *any or all the variables*, old and new, with or without t.

Let the equations connecting the old and new variables be, as before,

$$\frac{dP}{dy_i}=x_i,\ \frac{dP}{d\xi_i}=\eta_i, \qquad (83.)$$

with the condition that p, q, r, ... *are exempt from differentiation in forming these equations.*

Then take $\Psi=-\left(\frac{d(P)}{dt}\right)$, with the following signification; (1) $\frac{d(P)}{dt}$ denotes the differential coefficient of P with respect to t, so far as t is contained explicitly, and also through the variables in $p, q, r,\dots$; that is to say,

$$\frac{d(P)}{dt}=\frac{dP}{dt}+\frac{dP}{dp}\cdot p'+\frac{dP}{dq}\cdot q'+\dots$$

(where $p'=\frac{dp}{dt}+\frac{dp}{dx_1}x_1'+$&c. &c., but this substitution *is not to be made* at this stage); (2) $\left(\frac{d(P)}{dt}\right)$ denotes the result of substituting in the above expression the values of $y_1,\dots y_n$ in terms of $\xi_1,\dots\xi_n, \eta_1,\dots\eta_n, p, q, r,\dots$ from (83.), so far as $\frac{d(P)}{dt}$ contains $y_1,\dots y_n$ explicitly (*i. e.* not involved in $p, q, r,\dots$).

Lastly, take $$\Phi=(Z)+\Psi,$$

where (Z) denotes the result of substituting in Z the values of the old variables as given in terms of $\xi_1, \ldots \eta_1, \ldots p, q, r, \ldots$ from (82.). We shall then have

$$\xi_i' = \frac{d\Phi}{d\eta_i}, \quad \eta_i' = -\frac{d\Phi}{d\xi_i}, \quad \ldots\ldots\ldots\ldots \quad (84.)$$

where Φ is in general a function of

$$\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n, p, q, r, \ldots p', q', r', \ldots \text{ and } t;$$

and the differentiations with respect to ξ_i, η_i are performed only so far as those variables appear *explicitly* in Φ. But *after these differentiations*, we may introduce the actual values of $p, \ldots p', \ldots$ in terms of the variables and their differential coefficients. It is obvious that the original variables will not, *in general*, have been eliminated from the system (84.); but of course the elimination may be afterwards completed*. Similar considerations apply to that particular case of transformation which we have called a transformation of coordinates (art. 68.). We have then

$$P = (x_1)y_1 + \ldots + (x_n)y_n,$$

and the relations connecting $x_1 \ldots x_n$ with $\xi_1, \ldots \xi_n$, may contain $p, p, r, \ldots$; so that (x_1), &c. are functions of $p, q, r, \ldots$ as well as of $\xi_1, \ldots \xi_n$.

We might have deduced the preceding conclusions from the following simple consideration. Since $p, q, r, \ldots$ *are actually* functions of t, though *unknown* functions, we may imagine them to be *known*, and to be expressed explicitly in terms of t; and then the case resolves itself into that of art. 62, &c., so far as the demonstration is concerned. But as a doubt might possibly have arisen whether any fallacy was involved in the circumstance that $p, q, r, \ldots$ involve (when supposed to be expressed in terms of t) the *arbitrary constants of the problem itself*, it seemed best to refer to the original reasoning; the most important part of which is that contained (*mutatis mutandis*) in art. 6. (For the "mutanda" see the beginning of art. 63.) It is then apparent that this circumstance is perfectly immaterial with reference to the conclusions in question, though it may be important in other points of view.

72. This being premised, we will proceed to an example of transformation more interesting than the former, namely, the

Transformation from fixed to moving axes of coordinates.

Let x, y, z, u, v, w have the same signification as in art. 70, and let x, y, z, u, v, w be the new variables, where x, y, z are rectangular coordinates, referring to a system of moving axes of which the origin always coincides with that of the original fixed axes of x, y, z.

Let the direction cosines of the new (moving) axes with respect to the old be λ_0, μ_0, ν_0; λ_1, μ_1, ν_1; λ_2, μ_2, ν_2, thus†:

* The final equations in this case will not in general have the canonical form.

† I do not know who first used this convenient way of indicating the nine direction cosines by a diagram, but I first saw it in one of M. Lamé's works.

	x	y	z
x	λ_0	λ_1	λ_2
y	μ_0	μ_1	μ_2
z	ν_0	ν_1	ν_2

where λ_0, &c. are functions of t, which may be either given explicitly, or implicitly through the variables (see the last article).

The modulus of transformation P is found (art. 69.) by substituting for the variables x, y, z in the expression xu+yv+zw, their values in terms of x, y, z; we have therefore

$$\mathrm{P}=(\lambda_0 x+\lambda_1 y+\lambda_2 z)\mathrm{u}+(\mu_0 x+\mu_1 y+\mu_2 z)\mathrm{v}+(\nu_0 x+\nu_2 y+\nu_2 z)\mathrm{w};$$

and then the three equations

$$\frac{d\mathrm{P}}{dx}=u, \quad \frac{d\mathrm{P}}{dy}=v, \quad \frac{d\mathrm{P}}{dz}=w, \text{ give}$$

$$\left.\begin{aligned} u&=\lambda_0\mathrm{u}+\mu_0\mathrm{v}+\nu_0\mathrm{w} \\ v&=\lambda_1\mathrm{u}+\mu_1\mathrm{v}+\nu_1\mathrm{w} \\ w&=\lambda_2\mathrm{u}+\mu_2\mathrm{v}+\nu_2\mathrm{w}\end{aligned}\right\}, \text{ whence } \left\{\begin{aligned} \mathrm{u}&=\lambda_0 u+\lambda_1 v+\lambda_2 w \\ \mathrm{v}&=\mu_0 u+\mu_1 v+\mu_2 w \\ \mathrm{w}&=\nu_0 u+\nu_1 v+\nu_2 w\end{aligned}\right\} \quad \text{(85.)}$$

Also we have (see Theorem VIII. art. 64.), since P is to be considered to contain t explicitly through λ_0, &c., only,

$$\frac{d\mathrm{P}}{dt}=(\lambda'_0 x+\lambda'_1 y+\lambda'_2 z)\mathrm{u}+(\mu'_0 x+\mu'_1 y+\mu'_2 z)\mathrm{v}+(\nu'_0 x+\nu'_1 y+\nu'_2 z)\mathrm{w},$$

in which expression the values of u, v, w in terms of the new variables are to be substituted from (85.). Now if we put ω_0, ω_1, ω_2 for the angular velocities of the moving system of axes about the axes of x, y, z, respectively, so that

$$\omega_0=\lambda_2\lambda'_1+\mu_2\mu'_1+\nu_2\nu'_1=-(\lambda_1\lambda'_2+\mu_1\mu'_2+\nu_1\nu'_2), \text{ \&c.},$$

it will be immediately seen that the usual relations between the nine direction cosines enable us to put the result of the substitution in the following form:

$$\left(\frac{d\mathrm{P}}{dt}\right)=\omega_0(yw-zv)+\omega_1(zu-xw)+\omega_2(xv-yu).$$

The original differential equations

$$\mathrm{x}'=\frac{d\mathrm{Z}}{d\mathrm{u}}, \quad \mathrm{u}'=-\frac{d\mathrm{Z}}{d\mathrm{x}}, \text{ \&c}$$

are then transformed into

$$x'=\frac{d\Phi}{du}, \quad u'=-\frac{d\Phi}{dx}, \text{ \&c. (art. 64.)},$$

where $$\Phi=(\mathrm{Z})-\left(\frac{d\mathrm{P}}{dt}\right).$$

Introducing the above value of $\left(\frac{dP}{dt}\right)$, and omitting the brackets, we obtain for the system of transformed equations,

$$\left.\begin{aligned} x'&=\frac{dZ}{du}+\omega_2 y-\omega_1 z, & u'&=-\frac{dZ}{dx}+\omega_2 v-\omega_1 w\\ y'&=\frac{dZ}{dv}+\omega_0 z-\omega_2 x, & v'&=-\frac{dZ}{dy}+\omega_0 w-\omega_2 u\\ z'&=\frac{dZ}{dw}+\omega_1 x-\omega_0 y, & w'&=-\frac{dZ}{dz}+\omega_1 u-\omega_0 v \end{aligned}\right\} \quad \ldots\ldots \quad (86.)$$

in which Z is supposed to be expressed in terms of the new variables.

73. On the principles of the integration of this, and of transformed systems in general, I shall make some remarks hereafter. For the present, the following may be observed. If, in the transformation of the last article, we suppose the motion of the new axes *given*, then λ_0, &c., and therefore also ω_0, ω_1, ω_2, are given explicit functions of t. But if the motion of the new axes is only given by connecting it with the motion of the point m itself, then the above quantities are given functions of the variables and their differential coefficients.

The most interesting case of the latter kind is that in which the motion of the new axes is assumed to satisfy the equations

$$\frac{\omega_0}{x}=\frac{\omega_1}{y}=\frac{\omega_2}{z}, \quad \ldots\ldots\ldots \quad (\omega.)$$

which express the condition *that the instantaneous axis of rotation* (of the moving axes) *always coincides with the radius vector of the moving point* m*.

The radius vector traces, in fixed space, a certain conical surface. It also traces, with reference to the moving axes, another conical surface; and we might always assume as *one* of the conditions defining their motion, that this latter should be *any proposed surface*; that is, we might assume that the new coordinates x, y, z should always satisfy the equation $\varphi(x, y, z)=0$, φ representing any given homogeneous function. If to this last assumption we add the two conditions expressed by the formula (ω.), we further assume that *the conical surface traced by the radius vector with reference to the moving axes, rolls upon that traced in fixed space.*

Suppose, for example, we assume for the equation $\varphi(x, y, x)=0$, simply $z=0$. This, with the conditions (ω.), will express that the radius vector is always in the plane of xy, and that this plane rolls upon the conical surface traced by the radius vector in fixed space. We may then say that the plane of xy is the " plane of the orbit," and that the axes of xy, or any lines fixed with reference to them in their plane, are "fixed in the plane of the orbit†."

* See JACOBI's first letter to Professor HANSEN (CRELLE's Journal, vol. xlii. p. 21). This letter appears to refer to some unpublished (?) results of Professor HANSEN, which may possibly be similar to those of this article.

† The student of elementary treatises is, I believe, *always* left to find out for himself what this expression means, or ought to mean.

Now since we are supposing the equations to belong to a dynamical problem, we have $\mathrm{u}=m\mathrm{x}'$, $\mathrm{v}=m\mathrm{y}'$, $\mathrm{w}=m\mathrm{z}'$, and if we substitute on each side of these equations the values derived from (85.), art. 72, we find easily

$$u=m(x'+\omega_1 z-\omega_2 y),\ v=m(y'+\omega_2 x-\omega_0 z),\ w=m(z'+\omega_0 y-\omega_1 x),$$

relations which are true *on all suppositions as to the motion of the axes*; but the assumptions (ω.) reduce them to

$$u=mx',\ v=my',\ w=mz'.$$

The further assumption $z=0$, which involves $z'=0$, gives $w=0$, and also (by the equations (ω.)) $\omega_2=0$. Thus the equations (86.) are reduced to

$$x'=\frac{dZ}{du},\qquad u'=-\frac{dZ}{dx}$$

$$y'=\frac{dZ}{dv},\qquad v'=-\frac{dZ}{dy}$$

$$0=\frac{dZ}{dw},\qquad 0=-\frac{dZ}{dz}+\omega_1 u-\omega_0 v,$$

where in $\frac{dZ}{dw}$ and $\frac{dZ}{dz}$, z and w are to be put $=0$ after the differentiation. It is to be observed also, that the values of u, v, w above given reduce the three first of the equations (86.) to the form $u=m\frac{dZ}{du}$, $v=m\frac{dZ}{dv}$, $w=m\frac{dZ}{dw}$, from which it is evident that

$$Z=\frac{1}{2m}(u^2+v^2+w^2)-\mathrm{U},$$

where U is the original force-function, expressed in terms of the new variables. This depends only upon the conditions (ω.); but in the case now considered we have also $w=0$.

Let the origin of coordinates be the Sun; m a planet disturbed by another planet $m_{\prime}$ whose original coordinates are $\mathrm{x}_{\prime}, \mathrm{y}_{\prime}, \mathrm{z}_{\prime}$, and new coordinates $x_{\prime}, y_{\prime}, z_{\prime}$; also let $\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2=\mathrm{r}^2$, $x^2+y^2+z^2=r^2$, $\mathrm{x}_{\prime}^2+\mathrm{y}_{\prime}^2+\mathrm{z}_{\prime}^2=\mathrm{r}_{\prime}^2$, $x_{\prime}^2+y_{\prime}^2+z_{\prime}^2=r_{\prime}^2$, and $(\mathrm{x}_{\prime}-\mathrm{x})^2+(\mathrm{y}_{\prime}-\mathrm{y})^2+(\mathrm{z}_{\prime}-\mathrm{z})^2=\delta^2$; we have evidently $\mathrm{r}^2=r^2$, $\mathrm{r}_{\prime}^2=r_{\prime}^2$, and

$$(\mathrm{x}_{\prime}-\mathrm{x})^2+(\mathrm{y}_{\prime}-\mathrm{y})^2+(\mathrm{z}_{\prime}-\mathrm{z})^2=(x_{\prime}-x)^2+(y_{\prime}-y)^2+(z_{\prime}-z)^2,$$

and

$$\mathrm{xx}_{\prime}+\mathrm{yy}_{\prime}+\mathrm{zz}_{\prime}=xx_{\prime}+yy_{\prime}+zz_{\prime}.$$

We have then originally

$$\mathrm{U}=\frac{m\mu}{\mathrm{r}}+mm_1\left(\frac{1}{\delta}-\frac{\mathrm{xx}_{\prime}+\mathrm{yy}_{\prime}+\mathrm{zz}_{\prime}}{\mathrm{r}_{\prime}^3}\right),$$

where $\mu=m+$mass of Sun; and it is evident that this expression preserves the same form when expressed in terms of the new coordinates, and also (which is essential to the validity of what follows), that $\frac{d\mathrm{U}}{dx}$, &c. are the same whether the differentiation be performed before or after the substitution for $\mathrm{x}_{,}$ $\mathrm{y}_{,}$, $\mathrm{z}_{,}$, in terms of $x_{,}, y_{,}, z_{,}$. If then we put $z=0$ and

$$R=m_{,}\left(\frac{1}{\delta}-\frac{xx_{,}+yy_{,}}{r_{,}^{3}}\right),$$

we shall have, from the fourth and fifth of the transformed differential equations*,

$$\left.\begin{aligned} x''&=-\frac{\mu x}{r^3}+\frac{dR}{dx} \\ y''&=-\frac{\mu y}{r^3}+\frac{dR}{dy} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (87.)$$

from which it is evident, that, *assuming the motion of* $\mathrm{m}_{,}$ *to be known relatively to the new axes*, the variations of the four elements of the orbit of m which determine the dimensions of the orbit, its position relatively to lines *fixed in its own plane*, and the time of perihelion passage, will be expressed in terms of the differential coefficients of R in the same way as if the plane of the orbit were fixed. But the motion of the node of the orbit upon the fixed plane of xy, and its inclination to that plane, must be determined by means of the last of the differential equations, as follows: that equation gives

$$\omega_1 u-\omega_0 v=\left(\frac{dZ}{dz}\right)(z=0);$$

or if we put $-\Omega$ for the term multiplied by $mm_{,}$ in the value of U given above,

$$\omega_1 u-\omega_0 v=\frac{d}{dz}\left(-\frac{m\mu}{r}+\Omega\right),$$

and z is to be put $=0$ after the differentiation, which reduces the above to

$$\omega_1 u-\omega_0 v=\left(\frac{d\Omega}{dz}\right).$$

Let $\omega_0^2+\omega_1^2=\alpha^2$, so that α is the angular velocity of the plane of the orbit about the radius vector; then (observing that $u=mx'$, &c.) we have

$$\frac{\omega_0}{x}=\frac{\omega_1}{y}=\frac{\alpha}{r}=\frac{v\omega_0-u\omega_1}{m(xy'-x'y)},$$

whence
$$v\omega_0-u\omega_1=\frac{m\alpha}{r}(xy'-x'y)$$

$$=\frac{m\alpha}{r}\sqrt{\mu a(1-e^2)}\,\dagger,$$

and therefore
$$m\alpha=-\frac{r}{\sqrt{\mu a(1-e^2)}}\left(\frac{d\Omega}{dz}\right),$$

which gives α in terms of the four elements referred to above, and of t. And if we put ι for the inclination of the orbit to the plane of xy, ν for the longitude of the node

* These equations (87.) have been obtained in a different way by Mr. Bronwin. Camb. Math. Journ. vol. iv. p. 245.

† See below, art. 80.

referred to the axis of x, and β for the angle between the axis of x and the node, the usual formulæ of rotation give

$$\left.\begin{aligned} \iota' &= \omega_0 \cos\beta - \omega_1 \sin\beta \\ \nu' \sin\iota &= \omega_0 \sin\beta + \omega_1 \cos\beta \\ \beta' &= \omega_2 - \nu' \cos\iota \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (88.)$$

If in these expressions we put $\omega_0 = \frac{x}{r}\alpha$, $\omega_1 = \frac{y}{r}\alpha$, $\omega_2 = 0$, and call ϑ the angle between the radius vector and the node, so that $\omega_0 \cos\beta - \omega_1 \sin\beta = r\cos\vartheta$, $\omega_0 \sin\beta + \omega_1 \cos\beta = r\sin\vartheta$, we obtain finally

$$\iota' = \alpha\cos\vartheta, \quad \nu' = \alpha\frac{\sin\vartheta}{\sin\iota}, \quad \beta' = -\alpha\sin\vartheta\cot\iota,$$

in which the expression above given for α is to be substituted.

The actual value of $\left(\frac{d\Omega}{dz}\right)$ is $mm_, z_, \left(\frac{1}{r_,^3} - \frac{1}{\delta^3}\right)$, which (since $z_,$, $r_,$, &c. are supposed given in terms of t) may be expressed in terms of the four elements first mentioned, and t.

I propose to consider the transformation of the differential equations of the planetary theory in a more general manner in the following section. At present I shall add some remarks on normal transformations in general.

74. *Theorem.* If

$$x'_i = \frac{dZ}{dy_i}, \quad y'_i = -\frac{dZ}{dx_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89.)$$

be a system of $2n$ simultaneous differential equations, where

$$Z = f(x_1, y_1, x_2, y_2, \ldots p, q, r, \ldots, t),$$

and $p, q, r, \ldots$ are also explicit functions of x_1, &c., y_1, &c. and t, but are *exempt from differentiation* in taking the differential coefficients $\frac{dZ}{dx_i}, \frac{dZ}{dy_i}$; and if these equations be transformed by a normal substitution of new variables ξ_1, &c., η_1, &c. (art. 62, equation (73.)), then the transformed equations are, as in art. 64,

$$\xi'_i = \frac{d\Psi}{d\eta_i} + \frac{dZ}{d\eta_i}, \quad \eta'_i = -\frac{d\Psi}{d\xi_i} - \frac{dZ}{d\xi_i},$$

in which Z is expressed in terms of ξ_1, &c., η_1, &c., but the differentiations with respect to ξ_i, η_i are performed *before* the substitution of these variables in p, q, r, &c.; in other words, $p, q, r, \ldots$ are still to be exempt from differentiation in forming the differential equations.

This may be proved simply by repeating the reasoning of art. 64. The only difference is, that in the term

$$\Sigma_j \left\{ \frac{dZ}{dx_j} \frac{dx_j}{d\eta_i} + \frac{dZ}{dy_j} \frac{dy_j}{d\eta_i} \right\}$$

the differential coefficients $\frac{dZ}{dx_j}, \frac{dZ}{dy_j}$ are now taken only so far as Z contains x_j, y_j independently of p, q, r, &c.; and therefore the term represents the differential coefficient $\frac{dZ}{d\eta_i}$ taken so far as Z contains η_i independently of p, q, r, &c. The same reasoning applies to the corresponding term in the value of η'_i. The theorem is thus established.

It is evident that it may be combined with that given in art. 71, where other functions analogous to p, q, r, ... are introduced by the modulus of transformation P.

If we call the form of the system of differential equations (89.) *canonical* when the differentiations of Z with respect to x_1, &c., y_1, &c. are *total*, we might call it *pseudo-canonical* when Z contains functions of x_1, &c., y_1, which are exempt from differentiation in forming the differential equations.

In like manner, if we call a transformation of variables *normal*, when the differentiations of the modulus P (equations (73.), art. 62.) with respect to ξ_1, &c., y_1, &c. are total (as in art. 62.), we might call the transformation *pseudo-normal* when P contains functions of the variables which are exempt from differentiation in forming the equations of transformation (as in art. 71.).

Adopting these designations, we may enunciate the following general theorem of transformation:—

Theorem X.—If a *pseudo-canonical* system be transformed by a *normal* or *pseudo-normal* substitution, the transformed equations are also *pseudo-canonical*, and may be formed by the rules applying to normal transformations of canonical systems, provided that the functions which are originally exempt from differentiation with respect to the variables, be continued exempt to the end of the process; but if such functions occur in the modulus of transformation P, they are subject to total differentiation *with respect to* t in forming the term $\frac{dP}{dt}$. (See art. 71.)

[With respect to this theorem there is one important remark to be made. If u, v be any two functions of x_1, &c., y_1, &c. (with or without $p, q, r, \ldots$ and t), the equation

$$\Sigma_i\left(\frac{du}{dy_i}\frac{dv}{dx_i}-\frac{du}{dx_i}\frac{dv}{dy_i}\right)=\Sigma_i\left(\frac{du}{d\eta_i}\frac{dv}{d\xi_i}-\frac{du}{d\xi_i}\frac{dv}{d\eta_i}\right) \quad \text{(art. 63.)}$$

is now only true on condition that the substitution of the actual values of p, q, r, ... in terms of the variables be not performed till after all the differentiations.]

75. The theory of the variation of elements affords an interesting example of the theorem given in the last article. Consider the following system of differential equations,

$$x'_i=\frac{dZ}{dy_i}+\frac{d\Omega}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}-\frac{d\Omega}{dx_i}, \quad \ldots\ldots\ldots \quad (90.)$$

where in $\frac{dZ}{dx_i}$, $\frac{dZ}{dy_i}$ the differentiations are *total*, but Ω is supposed to contain functions of x_1, &c., y_1, &c., which are exempt from differentiation in forming the above equations. The system

$$x'_i=\frac{dZ}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}$$

is *canonical.* Let us assume then that a complete set of normal integrals $a_1, \dots a_n$, $b_1, \dots b_n$ of this latter system is known, so that we have

$$a_i = \phi_i(x_1, \text{\&c.}, y_1, \text{\&c.}, t), \quad b_i = \chi_i(x_1, \text{\&c.}, y_1, \text{\&c.}, t).$$

The assumption of these last equations to represent the solution of the complete system (90.) is simply a *transformation of variables*, (a_1, &c., b_1, &c. being the new variables); it is also a *normal transformation*, since the equations connecting the new and old variables may be put in the form (see Theorem VII. art. 49, and art. 62.)

$$\frac{dX}{dx_i} = y_i, \quad \frac{dX}{da_i} = b_i,$$

where X (the modulus of transformation) is a function of $x_1, \dots x_n, a_1, \dots a_n, t$. The function Ψ of art. 62 is now obtained by expressing $-\frac{dX}{dt}$ in terms of $a_1, \dots b_1, \dots$; but since Z is $-\frac{dX}{dt}$ expressed in terms of x_1, &c., y_1, &c., it follows that when Z is expressed in terms of the new variables a_1, &c., b_1, &c., it becomes identical with Ψ. Now if the process of art. 63 be followed, *mutatis mutandis*, it will be seen that in the present case we obtain

$$a_i' = \frac{d\Psi}{db_i} - \frac{dZ}{db_i} - \Sigma_j \left(\frac{d\Omega}{dy_j} \frac{dy_j}{db_i} \right),$$

in which expression the first two terms destroy one another, and the remaining term is evidently the differential coefficient of Ω with respect to b_i, taken so far as Ω contains b_i *independently of those functions which were exempt from differentiation* in forming the original differential equations (90.). Similar reasoning applies to the expression for b_i'.

As this result will be useful, I shall enunciate it separately as

Theorem XI.—If the original system of differential equations be formed by treating certain functions, $p, q, r, \dots$, contained in the disturbing function Ω, as exempt from differentiation with respect to x_1, &c., y_1, &c., the equations which determine the variations of any set of normal elements a_1, &c., b_1, &c.

$$a_i' = -\frac{d\Omega}{db_i}, \quad b_i' = \frac{d\Omega}{da_i}$$

on condition that $p, q, r, \dots$ be treated, in forming these equations, as exempt from differentiation with respect to a_1, &c., b_1, &c.

[It is important to recollect, that *after these equations are formed*, p, q, r, &c. are to be expressed in terms of a_1, &c., b_1, &c., and in the integration of the system a_1, &c., b_1, &c. are to be treated indiscriminately as variables, whether they originally entered through $p, q, r, \dots$ or not].

The Theorem XI. may also be immediately obtained from the general equations (E.) of art. 52 (in which it is to be remembered that Z *includes* the disturbing function). The above method of deducing it is given as an additional illustration of the general theory of transformation.

If, instead of the normal elements a_1, &c., b_1, &c., we employ any other elements, c_1, c_2, &c., which can be expressed as functions of the former, the formula (69.) of art. 54 will still be applicable, with the condition that p, q, r, &c. are exempt from differentiation with respect to c_1, c_2, &c.

SECTION VII.—*On the Differential Equations of the Planetary Theory.*

76. The differential equations which determine the motions of a system of mutually attracting material points *relatively to one of them*, do not, as is well known, naturally present themselves under the canonical form (I.), art. 49. It is possible indeed to reduce them, by different artifices, to that form; but it seems doubtful whether any practical advantage is gained by doing so. When the ordinary method is followed in the case of a planetary system referred to the sun, there is a distinct disturbing function for each planet; but it is easily seen that the usual expressions for the variations of the elements hold good, not merely for each planet on the hypothesis that the motions of the rest are *known*, but as a complete and rigorous set of simultaneous differential equations involving all the elements of all the orbits, and their differential coefficients with respect to t (and containing of course also t explicitly). It does not appear that we are practically farther from the attainment of the rigorous integration of this system, than we should be if it had the *canonical form*, as it might be made to have if it were derived from an original system of that form; and so far as the development of the disturbing functions is concerned, the most troublesome part of them, which is that depending on the mutual distances of the planets, is not likely to be got rid of by any conceivable artifice.

However this may be, all that I propose to do in the present section is to take the original differential equations in their ordinary form referred to rectangular axes passing through the sun and parallel to fixed directions, and then to exhibit in a general manner the effect of a transformation to new rectangular axes, still passing through the sun, but changing their directions in space according to any arbitrary law.

77. Let M be the mass of the sun, and m, $m_{\prime}$, $m_{\prime\prime}$, &c., the masses of the planets; and put $\mathrm{M}+m=\mu$, $\mathrm{M}+m_{\prime}=\mu_{\prime}$, $\mathrm{M}+m_{\prime\prime}=\mu_{\prime\prime}$, ... $\mathrm{M}+m_{(i)}=\mu_{(i)}$. And, referred to the original axes, let x, y, z be the coordinates of m, $\mathrm{x}_{\prime}$, $\mathrm{y}_{\prime}$, $\mathrm{z}_{\prime}$ of $m_{\prime}$, ..., $\mathrm{x}_{(i)}$, $\mathrm{y}_{(i)}$, $\mathrm{z}_{(i)}$ of $m_{(i)}$. Also let R, $\mathrm{R}_{\prime}$, ... $\mathrm{R}_{(i)}$ have their usual significations, so that

$$\mathrm{R}=\Sigma\left\{\frac{m_{(i)}}{\left((\mathrm{x}_{(i)}-\mathrm{x})^2+(\mathrm{y}_{(i)}-\mathrm{y})^2+(\mathrm{z}_{(i)}-\mathrm{z})^2\right)^{\frac{1}{2}}}-\frac{m_{(i)}(\mathrm{xx}_{(i)}+\mathrm{yy}_{(i)}+\mathrm{zz}_{(i)})}{(\mathrm{x}^2_{(i)}+\mathrm{y}^2_{(i)}+\mathrm{z}^2_{(i)})^{\frac{3}{2}}}\right\},$$

the summation extending to all the planets except m.

Then if we put $\mathrm{x}^2_{(i)}+\mathrm{y}^2_{(i)}+\mathrm{z}^2_{(i)}=\mathrm{r}^2_{(i)}$, the original differential equations of the second order are such as $\mathrm{x}''_{(i)}+\mu\frac{\mathrm{x}_{(i)}}{\mathrm{r}^3_{(i)}}=\frac{d\mathrm{R}_{(i)}}{d\mathrm{x}_{(i)}}$. Let $\mathrm{u}_{(i)}$, $\mathrm{v}_{(i)}$, $\mathrm{w}_{(i)}$ be the variables conjugate to

$x_{(i)}, y_{(i)}, z_{(i)}$. Then instead of the original system of differential equations of the second order, we have the following system of the first order:

$$x'_{(i)}=\frac{dZ_{(i)}}{du_{(i)}},\quad y'_{(i)}=\frac{dZ_{(i)}}{dv_{(i)}},\quad z'_{(i)}=\frac{dZ_{(i)}}{dw_{(i)}}$$

$$u'_{(i)}+\frac{dZ_{(i)}}{dx_{(i)}}=0,\quad v'_{(i)}+\frac{dZ_{(i)}}{dy_{(i)}}=0,\quad w'_{(i)}+\frac{dZ_{(i)}}{dz_{(i)}}=0,$$

in which $$Z_{(i)}=\frac{1}{2m_{(i)}}(u^2_{(i)}+v^2_{(i)}+w^2_{(i)})-m_{(i)}\left(\frac{\mu_{(i)}}{r_{(i)}}+R_{(i)}\right).$$

These equations are not of the canonical form, because $R_{(i)}$ is not the same for all the planets. But it is easy to put them in the *pseudo-canonical* form (art. 74.), a process which is not *necessary*, but saves trouble by bringing them under the operation of the general rules of transformation established in former articles.

In fact, if we take

$$Z=\Sigma\left(\frac{u^2_{(i)}+v^2_{(i)}+w^2_{(i)}}{2m_{(i)}}-\frac{m_{(i)}\mu_{(i)}}{r_{(i)}}\right)$$

$$Q=\Sigma m_{(i)}m_{(j)}\left(\frac{x_{(i)}\bar{x}_{(j)}+y_{(i)}\bar{y}_{(j)}+z_{(i)}\bar{z}_{(j)}}{\bar{r}^2_{(j)}}+\frac{\bar{x}_{(i)}x_{(j)}+\bar{y}_{(i)}y_{(j)}+\bar{z}_{(i)}z_{(j)}}{\bar{r}^2_{(i)}}\right.$$
$$\left.-\frac{1}{((x_{(i)}-x_{(j)})^2+(y_{(i)}-y_{(j)})^2+(z_{(i)}-z_{(j)})^2)^{\frac{1}{2}}}\right),$$

where the summation in the first term extends to all the planets, and in the second to all their binary combinations, and a horizontal line placed over any letter indicates *exemption from differentiation*, we shall have

$$x'_{(i)}=\frac{dZ}{du_{(i)}}+\frac{dQ}{du_{(i)}},\quad u'_{(i)}=-\frac{dZ}{dx_{(i)}}-\frac{dQ}{dx_{(i)}}, \quad \ldots \ldots \quad (91.)$$

with similar equations for $y'_{(i)}, z'_{(i)}, v'_{(i)}, w'_{(i)}$.

[The terms $\frac{dQ}{du_{(i)}}$ &c. are only written for the sake of uniformity, being really $=0$, since Q does not involve $u_{(i)}, v_{(i)}, w_{(i)}$.]

In these equations Z and Q are *the same for the whole system*, and the differentiations of Z are *total*; but those of Q are restricted to the quantities not marked by the horizontal line, so that $\frac{dQ}{dx_{(i)}}$ is really the same thing as $-m_{(i)}\frac{dR_{(i)}}{dx_{(i)}}$.

78. Let us now refer the whole system to new (moving) rectangular axes, whose position at any instant with respect to the original axes is defined, as in art. 72, by the variable direction-cosines λ_0, &c. Let $x_{(i)}, y_{(i)}, z_{(i)}$ be the new coordinates, and $u_{(i)}, v_{(i)}, w_{(i)}$ the new variables conjugate to them. The transformation will be effected, as in art. 72, by taking for the modulus

$$P=\Sigma\Big((\lambda_0x_{(i)}+\lambda_1y_{(i)}+\lambda_2z_{(i)})u_{(i)}+(\mu_0x_{(i)}+\mu_1y_{(i)}+\mu_2z_{(i)})v_{(i)}+(\nu_0x_{(i)}+\nu_1y_{(i)}+\nu_2z_{(i)})w_{(i)}\Big),$$

and the result will be as follows: put

$$\Omega=Q+\bar{\omega}_0\Sigma(z_{(i)}v_{(i)}-y_{(i)}w_{(i)})+\bar{\omega}_1\Sigma(x_{(i)}w_{(i)}-z_{(i)}u_{(i)})+\bar{\omega}_2\Sigma(y_{(i)}u_{(i)}-x_{(i)}v_{(i)})$$

(where Q is expressed in terms of the new variables, and ω_0, ω_1, ω_2 (the angular velocities of the moving system of axes about the three moving axes themselves) are marked with the horizontal line to show that these quantities are exempt from differentiation in forming the following system of differential equations, though they may be functions of the variables); then the system (91.) is transformed into

$$x'_{(i)}=\frac{dZ}{du_{(i)}}+\frac{d\Omega}{du_{(i)}},\quad u'_{(i)}=-\frac{dZ}{dx_{(i)}}-\frac{d\Omega}{dx_{(i)}}, \quad \ldots \ldots \quad (92.)$$

with similar equations for y'_i, &c.

In these equations Z is to be expressed in terms of the new variables; and it is evident from the original form of Z and Q, that when so expressed, these two quantities are the same functions of the new variables that they were of the old, and involve (see art. 74.) the quantities exempt from differentiation in the same way*. Thus the transformed system (92.) contains no terms *explicitly depending upon the motion of the axes*, except those introduced by the three terms multiplied by ω_0, ω_1, ω_2 in the value of Ω given above; and the addition of these terms constitutes the only difference between the form of the old and of the new system.

79. We may now apply the method of the variation of elements to the system (92.) as follows:—

The system obtained by omitting the disturbing function Ω, namely,

$$\left.\begin{array}{lll} x'_{(i)}=\frac{dZ}{du_{(i)}}, & y'_{(i)}=\frac{dZ}{dv_{(i)}}, & z'_{(i)}=\frac{dZ}{dw_{(i)}} \\ u'_{(i)}+\frac{dZ}{dx_{(i)}}=0, & v'_{(i)}+\frac{dZ}{dy_{(i)}}=0, & w'_{(i)}+\frac{dZ}{dz_{(i)}}=0 \end{array}\right\} \quad \ldots \ldots \quad (93.)$$

is *canonical*, and consists simply of the aggregate of the equations representing, for each planet, undisturbed elliptic motion about the sun † (relatively to the new axes of coordinates).

The integrals of these equations may therefore be expressed in any of the usual forms. We will suppose that the elements chosen are

$$a,\ e,\ \varpi,\ (\varepsilon),\ \iota,\ \nu,$$

with significations corresponding to those given to the same symbols in art. 55. These letters unaccented will apply to the planet m, and $a_{,}$, $e_{,}$, &c., $a_{,,}$, $e_{,,}$, &c., $a_{(i)}$, $e_{(i)}$, &c., to the planets $m_{,}$, $m_{,,}$, $m_{(i)}$, &c.

The *definitions* of the elements a, e, ϖ, &c. are their expressions in terms of the six

* Since the direction cosines λ_0, &c. are exempt from differentiation in forming the equations connecting the old and new variables from the modulus P, they continue exempt throughout. (Theorem X. art. 74.) Hence we have

$$\mathrm{x}\bar{\mathrm{x}}_{,}+\mathrm{y}\bar{\mathrm{y}}_{,}+\mathrm{z}\bar{\mathrm{z}}_{,}=(\bar{\lambda}_0x+\bar{\lambda}_1y+\bar{\lambda}_2z)(\overline{\lambda_0x_{,}+\lambda_1y_{,}+\lambda_2z_{,}})+\&c.=x\bar{x}_{,}+y\bar{y}_{,}+z\bar{z}_{,},$$

and similarly for the rest.

† $Z=\Sigma\left(\frac{u^2+v^2+w^2}{2m}-\frac{\mu m}{r}\right)$, the summation extending to all the planets.

variables x, y, z, u, v, w *, and t, and the same expressions continue to be their definitions when they become the variable elements of the disturbed motion.

Now the general formulæ for the variations of the set of elements here chosen have already been given in art. 55; for it is evident that the process in the present case would merely be a repetition, for each planet, of the process there employed. Here however we are to use in every case the disturbing function Ω given in art. 78; but if we observe the effect of the *marks of exemption*, it will be evident that, for the planet m, the only effective part of Ω is

$$\bar{\omega}_0(zv-yw)+\bar{\omega}_1(xw-zu)+\bar{\omega}_2(yu-xv)-m\mathrm{R};$$

and similarly the effective part of Ω for any other planet will be given by suffixing the corresponding number of accents to x, y, z, u, v, w, m, R.

Now if we put A_0, A_1, A_2 for the terms multiplied respectively by ω_0, ω_1, ω_2 in the above expression, we have, by the definitions of the elements,

$$(\mathrm{A}_0^2+\mathrm{A}_1^2+\mathrm{A}_2^2)^{\frac{1}{2}}=m\sqrt{\mu a(1-e^2)}$$

$$\mathrm{A}_2=-m\sqrt{\mu a(1-e^2)}.\cos\iota,\quad \mathrm{A}_1=m\sqrt{\mu a(1-e^2)}.\cos\nu\sin\iota$$

$$\mathrm{A}_0=-m\sqrt{\mu a(1-e^2)}.\sin\nu\sin\iota,$$

so that the disturbing function, so far as m is concerned, becomes

$$-m\{\mathrm{R}+\sqrt{\mu a(1-e^2)}.(\bar{\omega}_0\sin\nu\sin\iota-\bar{\omega}_1\cos\nu\sin\iota+\bar{\omega}_2\cos\iota)\}. \quad \ldots \quad (94.)$$

Consequently, since the expressions in art. 55 were obtained by taking $-m$R for the disturbing function, we have merely to add to them the additional terms derived from the part added to R in (94.). Performing the differentiations, and omitting afterwards the symbols of exemption over ω_0, ω_1, ω_2 which cease to be of any use, we obtain, after obvious reductions, the following simple results: if $\frac{\partial a}{dt}$, $\frac{\partial e}{dt}$, &c. represent those parts of the differential coefficients of a, e, &c., with respect to t, which depend upon the motion of the axes of coordinates, then

$$\left.\begin{aligned}
&\frac{\partial a}{dt}=0,\quad \frac{\partial e}{dt}=0\\
&\frac{\partial(\varepsilon)}{dt}=\frac{\partial\varpi}{dt}=\tan\frac{\iota}{2}(\omega_1\cos\nu-\omega_0\sin\nu)-\omega_2\\
&\frac{\partial\iota}{dt}=-(\omega_0\cos\nu+\omega_1\sin\nu)\\
&\frac{\partial\nu}{dt}=-\omega_2+\cot\iota\,(\omega_0\sin\nu-\omega_1\cos\nu)
\end{aligned}\right\} \quad \ldots \quad (95.)$$

where it is evident that we may write ε instead of (ε) (see art. 55.).

* *Not* their expressions in terms of x, y, z, x', y', z'; for though these are equivalent in the undisturbed equations, they are not here equivalent in the disturbed equations, and therefore the general theory, which assumes the former mode of expression, is not here applicable to the latter.

The complete variations of the elements are then found by adding to the terms just written the expressions given in art. 25.

It is easily seen that the expressions (95.) might have been deduced from geometrical considerations alone, if we had been at liberty to assume beforehand that the *mechanical* and *geometrical* parts of the variations might be calculated separately; the former as if the axes were at rest, and the latter as if there were no disturbing forces. It would not, I believe, be difficult to establish by *à priori* and simple reasoning the validity of such an assumption, and then the above results would only serve as a verification of the method which has been employed to obtain them.

80. In order however that no obscurity may rest upon the interpretation of the formulæ obtained in the last articles, it is necessary to consider the physical (or rather geometrical) meaning of the elements a, e, &c., which we have so far only defined by means of their expressions in terms of the variables x, y, z, u, &c., and to ascertain what relation they bear to the elements similarly defined by means of the original variables x, y, &c., which refer to axes whose directions are invariable.

The relations between the variables ((85.), art. 72.) give immediately

$$x^2+y^2+z^2=\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2$$

$$u^2+v^2+w^2=\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2$$

$$ux+vy+zw=\mathrm{ux}+\mathrm{vy}+\mathrm{zw}\,;$$

and if we put

$$yw-zv=A,\quad zu-xw=B,\quad xv-yu=C$$

$$\mathrm{yw}-\mathrm{zv}=\mathrm{A},\quad \mathrm{zu}-\mathrm{xw}=\mathrm{B},\quad \mathrm{xv}-\mathrm{yu}=\mathrm{C},$$

we find, by virtue of the relations $\mu_1\nu_2-\nu_1\mu_2=\lambda_0$, &c., the following equations:

$$A=\lambda_0\mathrm{A}+\mu_0\mathrm{B}+\nu_0\mathrm{C}$$

$$B=\lambda_1\mathrm{A}+\mu_1\mathrm{B}+\nu_1\mathrm{C}$$

$$C=\lambda_2\mathrm{A}+\mu_2\mathrm{B}+\nu_2\mathrm{C}$$

and

$$A^2+B^2+C^2=\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2.$$

Now A$(=\mathrm{yw}-\mathrm{zv}=m(\mathrm{yz}'-\mathrm{zy}'))$ is the projection on the plane of yz of the areal velocity of m (relative to fixed space) multiplied by the mass, and B, C have analogous meanings; hence it is evident from the above equations that A, B, C are the projections on the three moving coordinate planes of yz, zx, xy of the absolute areal velocity of m relative to fixed space, multiplied by the mass. (The projections of the areal velocity *relative to the moving axes* would be $yz'-zy'$, &c., which are not proportional to $yw-zv$, &c., since u, v, w are not the same as mx', my', mz', except on a particular hypothesis as to the motion of the axes. See art. 73.).

Inasmuch as the definitions of the elements a, e, ι involve the variables only in the forms $x^2+y^2+z^2$, $u^2+v^2+w^2$, A, B, C, it follows that these three elements are respectively the *semiaxes, excentricity, and inclination to the plane of xy, of the absolute osculating ellipse of the orbit in fixed space.* Thus the instantaneous ellipse, relatively to the moving axes, is of the same dimensions and in the same plane as the true osculating ellipse; and it only remains to show that it coincides with the latter in position, for which purpose we must prove that it *touches the true orbit.* (It does *not* in general touch the relative orbit, because mx', my', mz' are not in general the same functions of the elements and t that u, v, w are.)

81. If we suppose the coordinates x, y, z of m expressed in terms of the elements a, e, &c. and t, and denote by $\frac{dx}{dt}$, &c. their differential coefficients taken so far as t appears explicitly in these expressions, then $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are proportional to the direction cosines, relatively to the axes of x, y, z, of the tangent to the (relative) instantaneous ellipse. And therefore the direction cosines of this same tangent referred to the fixed axes of x, y, z, are proportional respectively to

$$\lambda_0\frac{dx}{dt}+\lambda_1\frac{dy}{dt}+\lambda_2\frac{dz}{dt},\quad \mu_0\frac{dx}{dt}+\mu_1\frac{dy}{dt}+\mu_2\frac{dz}{dt},\quad \nu_0\frac{dx}{dt}+\nu_1\frac{dy}{dt}+\nu_2\frac{dz}{dt}.$$

On the other hand, the direction cosines of the tangent to the *absolute* instantaneous ellipse, referred to the fixed axes, are proportional to $\frac{d\mathrm{x}}{dt}, \frac{d\mathrm{y}}{dt}, \frac{d\mathrm{z}}{dt}$ (the differential coefficients of x, y, z, taken so far as t appears explicitly in the expressions for those variables in terms of the elements of the absolute ellipse). The identity of the two tangents will therefore be established, if we can show that

$$\frac{d\mathrm{x}}{dt}=\lambda_0\frac{dx}{dt}+\lambda_1\frac{dy}{dt}+\lambda_2\frac{dz}{dt}, \text{ \&c.}$$

Now $\frac{dx}{dt}$ is that part of x' which does not depend upon the disturbing function; i. e. (equations (92.), art. 78.) we have identically

$$\frac{dx}{dt}=\frac{dZ}{du},\quad \frac{dy}{dt}=\frac{dZ}{dv},\quad \frac{dz}{dt}=\frac{dZ}{dw};$$

and, in like manner,

$$\frac{d\mathrm{x}}{dt}=\frac{dZ}{d\mathrm{u}},\quad \frac{d\mathrm{y}}{dt}=\frac{dZ}{d\mathrm{v}},\quad \frac{d\mathrm{z}}{dt}=\frac{dZ}{d\mathrm{w}},$$

where Z is the same as before, but expressed in terms of x, y, z, u, v, w, instead of x, y, z, u, v, w. Now let the latter set of variables be expressed in terms of the former by the formulæ of art. 72, and we have

$$\frac{dZ}{d\mathrm{u}}=\frac{dZ}{du}\frac{du}{d\mathrm{u}}+\frac{dZ}{dv}\frac{dv}{d\mathrm{u}}+\frac{dZ}{dw}\frac{dw}{d\mathrm{u}}, \text{ \&c.},$$

but

$$\frac{du}{d\mathrm{u}}=\lambda_0,\quad \frac{dv}{d\mathrm{u}}=\lambda_1,\quad \frac{dw}{d\mathrm{u}}=\lambda_2 \text{ ((85.), art. 72.), \&c.},$$

and therefore

$$\frac{d\mathrm{x}}{dt}=\lambda_0\frac{dx}{dt}+\lambda_1\frac{dy}{dt}+\lambda_2\frac{dz}{dt}, \text{ \&c.},$$

as was to be proved.

82. It follows, then, that the mode of treating the problem adopted in the preceding articles is equivalent to representing the motion of each planet by means of the true osculating ellipse of its actual orbit (relatively to the sun) in fixed space.

The definitions of all the elements (relative to the moving axes) in terms of the six new variables x, y, z, u, v, w, have the same form as those of the corresponding elements (relative to the fixed axes) in terms of x, y, z, u, v, w. The two relative elements a, e are the same as the corresponding *absolute* elements; ι is the inclination of the plane of the ellipse to the moving plane of xy, and ν the longitude of the node reckoned from the axis of x; and since the place of the body in the ellipse is evidently the same, the relations between the remaining elements ϖ and (ε) (or ε) and the corresponding *absolute elements* are purely geometrical.

Comparing these results with those of art. 79, we see that the independence of the formulæ for the mechanical and geometrical variations of the elements of the true osculating ellipse is completely established.

83. In all that precedes, the three variables ω_0, ω_1, ω_2 (the angular velocities of the system of moving axes about the axes themselves) are entirely arbitrary; they may be either explicit functions of t, involving only determinate constants, or they may depend in any way upon the relative or absolute elements of the orbits of any or all the planets, and their differential coefficients with respect to t. In the case in which the expressions for ω_0, ω_1, ω_2 involve only the *relative* elements, when these expressions are introduced in the formulæ (95.), art. 79, and these formulæ completed by the addition of the terms in art. 55, and when the corresponding sets of equations are formed for each planet, we obtain a set of simultaneous differential equations involving all the elements of all the orbits, and their differential coefficients with respect to t. The integration of these equations would determine all the elements as functions of t, and thus the motion of all the planets, relatively to the axes of coordinates, would be known. Lastly, the motion of the whole system, relatively to fixed space, would be found by integrating the system of equations

$$\left.\begin{aligned} \mathrm{L}' &= \omega_0 \cos \mathrm{X} - \omega_1 \sin \mathrm{X} \\ \mathrm{N}' \sin \mathrm{L} &= \omega_0 \sin \mathrm{X} + \omega_1 \cos \mathrm{X} \\ \mathrm{X}' &= \omega_2 - \mathrm{N}' \cos \mathrm{L} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96.)$$

where ω_0, ω_1, ω_2 are now given functions of t, and L is the inclination of the plane of xy to that of xy, N is the longitude of the ascending* node of the plane of xy, reckoned from the axis of x, and X is the longitude of the axis of x, reckoned upon the plane of xy, *from the node*, in the direction of positive rotation.

In the case in which ω_0, ω_1, ω_2 cannot be expressed in terms of the *relative* elements, the integrations which determine the relative motion of the system cannot be separated from those which determine the position of the axes in fixed space; but the equations (96.) must be considered simultaneously with the other differential equations of the problem.

* *Ascending* relatively to a positive rotation, *i. e.* from x to y.

Application to the Problem of Three Bodies.

84. I propose to exemplify the preceding principles by applying them to the transformation of the equations which determine the motion of three mutually attracting bodies, considered as material points. Let them be called the sun and two planets, and let the origin of coordinates be placed at the sun, and the notation be the same as in arts. 77–79, so that M, m, $m_{,}$, are the three masses, $M+m=\mu$, $M+m_{,}=\mu_{,}$, &c. Let the elements be chosen as in art. 55, and longitudes be measured, as before, along the plane of xy (from the axis of x) as far as the node, and then along the plane of the orbit.

Putting for convenience $mR=\Omega$, $m_{,}R_{,}=\Omega_{,}$, we have

$$\Omega=mm_{,}\{(r^2+r_{,}^2-2rr_{,}\cos\chi)^{-\frac{1}{2}}-rr_{,}^{-2}\cos\chi\}$$
$$\Omega_{,}=mm_{,}\{(r^2+r_{,}^2-2rr_{,}\cos\chi)^{-\frac{1}{2}}-r_{,}r^{-2}\cos\chi\},$$

where χ is the angle between the two radii vectores. Let I be the mutual inclination of the planes of the two orbits, and let the angular distances, on the planes of the orbits, between their ascending nodes on the plane of xy and their line of intersection, be respectively υ, $\upsilon_{,}$; so that $\theta-\upsilon-\nu$, $\theta_{,}-\upsilon_{,}-\nu_{,}$ are the angular distances of the two radii vectores from the line of intersection (which we may call, simply, *the line of nodes*). We shall then have

$$\left.\begin{array}{l}\cos\chi=\cos(\theta-\upsilon-\nu)\cos(\theta_{,}-\upsilon_{,}-\nu_{,})+\sin(\theta-\upsilon-\nu)\sin(\theta_{,}-\upsilon_{,}-\nu_{,})\cos I\\ \cos I=\cos\iota\cos\iota_{,}+\sin\iota\sin\iota_{,}\cos(\nu_{,}-\nu)\,*\end{array}\right\} \quad . \quad (97.)$$

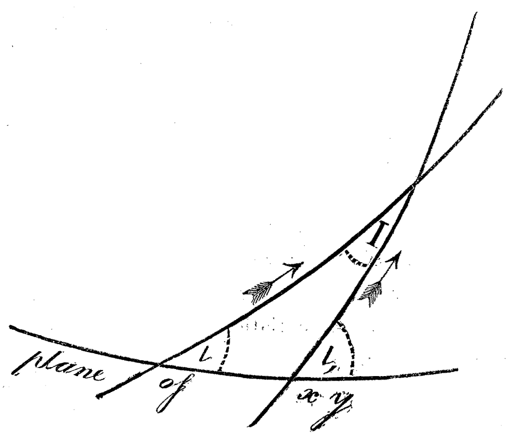
and υ, $\upsilon_{,}$ are functions of ι, $\iota_{,}$, $\nu_{,}-\nu$, determined by the equations

$$\left.\begin{array}{l}\cot\upsilon\sin(\nu_{,}-\nu)=-\cot\iota_{,}\sin\iota+\cos(\nu_{,}-\nu)\cos\iota\\ \cot\upsilon_{,}\sin(\nu_{,}-\nu)=\cot\iota\sin\iota_{,}+\cos(\nu_{,}-\nu)\cos\iota_{,}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (98.)$$

Now considering Ω as expressed, on the one hand, in terms of r, $r_{,}$, θ, $\theta_{,}$, ν, $\nu_{,}$, ι, $\iota_{,}$, and on the other in terms of all the elements and t, we have rigorously, as may easily be proved in the usual way †,

$$\frac{d\Omega}{d\theta}=\frac{d\Omega}{d\varepsilon}+\frac{d\Omega}{d\varpi}, \quad \frac{d\Omega}{d\theta_{,}}=\frac{d\Omega}{d\varepsilon_{,}}+\frac{d\Omega}{d\varpi_{,}},$$

* The arrangement referred to here and in the following articles will be made clear by the accompanying diagram.



† Since the values of r and θ in terms of the elements and t are of the forms

$$r=f(\int ndt+\varepsilon-\varpi), \quad \theta=\int ndt+\varepsilon+\phi(\int ndt+\varepsilon-\varpi),$$

we have $$\frac{dr}{d\varepsilon}+\frac{dr}{d\varpi}=0, \quad \frac{d\theta}{d\varepsilon}+\frac{d\theta}{d\varpi}=1;$$

from which the equations in the text follow immediately. It may be as well to remind the reader who may happen to recollect the note to the Astronomer Royal's tract on the Planetary Theory (p. 91, ed. 1831), that that note refers to a different way of measuring longitudes.

with similar equations for $\Omega_{\prime}$. In what follows, I shall, as an abridgment, employ the symbol E to denote the operation $\frac{d}{d\varepsilon}+\frac{d}{d\varpi}$, and in like manner I shall put $E_{\prime}$ for $\frac{d}{d\varepsilon_{\prime}}+\frac{d}{d\varpi_{\prime}}$, so that

$$E\Omega=\frac{d\Omega}{d\varepsilon}+\frac{d\Omega}{d\varpi}, \text{ \&c.} \quad \ldots \ldots \quad (99.)$$

Since r and θ, when expressed in terms of the elements and t, do not contain ι, $\iota_{\prime}$, ν, $\nu_{\prime}$, we have

$$\frac{d\Omega}{d\nu}=\frac{d\Omega}{d\cos\chi}\cdot\frac{d\cos\chi}{d\nu},\quad \frac{d\Omega}{d\iota}=\frac{d\Omega}{d\cos\chi}\cdot\frac{d\cos\chi}{d\iota}$$

and

$$\frac{d\cos\chi}{d\nu}=-\left(1+\frac{dv}{d\nu}\right)\frac{d\cos\chi}{d\theta}-\frac{dv_{\prime}}{d\nu}\frac{d\cos\chi}{d\theta_{\prime}}$$
$$+\sin\iota\sin\iota_{\prime}\sin(\theta-v-\nu)\sin(\theta_{\prime}-v_{\prime}-\nu_{\prime})\sin(\nu_{\prime}-\nu),$$

in which expression the values of $\frac{dv}{d\nu}$, $\frac{dv_{\prime}}{d\nu_{\prime}}$ are to be obtained by differentiating the equations (98.). In like manner we find

$$\frac{d\cos\chi}{d\iota}=-\frac{dv}{d\iota}\frac{d\cos\chi}{d\theta}-\frac{dv_{\prime}}{d\iota}\frac{d\cos\chi}{d\theta_{\prime}}$$
$$+\sin(\theta-v-\nu)\sin(\theta_{\prime}-v_{\prime}-\nu_{\prime})\big(-\sin\iota\cos\iota_{\prime}+\cos\iota\sin\iota_{\prime}\cos(\nu_{\prime}-\nu)\big).$$

Analogous expressions may be found for $\frac{d\Omega_{\prime}}{d\nu_{\prime}}$, $\frac{d\Omega_{\prime}}{d\iota_{\prime}}$.

85. Hitherto we have assumed nothing concerning the motion of the axes of coordinates. Let us now however take as a first assumption that *the plane of xy shall always pass through the line of nodes.* This implies the conditions*

$$\nu_{\prime}=\nu, \quad v=0, \quad v_{\prime}=0,$$

and consequently $I=\iota_{\prime}-\iota$.

Introducing these conditions in the expressions at the end of the last article, and

* The legitimacy of the following processes will be apparent to the reader who shall have followed the general reasoning of former articles, though probably not to others. In either case it may be useful here briefly to recapitulate the principles now to be applied. The results of art. 79, and in particular the expressions (94.), (95.), were established independently of any assumption as to the values of ω_0, ω_1, ω_2, which are perfectly arbitrary. We are therefore at liberty to assume that ω_0, ω_1, ω_2 are such functions of t that any three functions of the variables shall constantly $=0$. Thus the first assumption made in the text is that $\nu_{\prime}-\nu$ shall constantly $=0$. But since such assumptions may cause (as in this case) certain elements to disappear from the expression of Ω, it is necessary to perform the differentiations of Ω with respect to such elements *first*; the differential coefficients may or may not vanish, on afterwards introducing the assumptions; if not, we find expressions for them in terms of differential coefficients with respect to *other elements which do not disappear*; so that instead of differentiations which ought to be performed *before* the assumed conditions are introduced, we have finally only such as may be performed *afterwards*. The expression "disappear" does not, it must be observed, necessarily mean the same as "vanish." Thus the condition $\nu_{\prime}-\nu=0$, causes ν and $\nu_{\prime}$ both to *disappear* from Ω, but does not of course imply that they both *vanish*.

observing that $\frac{d\Omega}{d\theta}=\mathrm{E}\Omega$, &c. (see equation (99.)), we have evidently

$$\frac{d\Omega}{d\nu}=-\left(1+\left(\frac{dv}{d\nu}\right)\right)\mathrm{E}\Omega-\left(\frac{dv_{\prime}}{d\nu}\right)\mathrm{E}_{\prime}\Omega$$

$$\frac{d\Omega}{d\iota}=-\left(\frac{dv}{d\iota}\right)\mathrm{E}\Omega-\left(\frac{dv_{\prime}}{d\iota}\right)\mathrm{E}_{\prime}\Omega-\frac{d\Omega}{d\mathrm{I}};$$

and it only remains to find the values of $\left(\frac{dv}{d\nu}\right)$, &c., namely, the values of $\frac{dv}{d\nu}$, &c., which correspond to the assumed conditions $\nu_{\prime}-\nu=0,\ v=v_{\prime}=0$.

Now in any spherical triangle of which a, b, c are the sides, and A, B, C the opposite angles, if a be considered as a function of c, A, B, by virtue of the equation

$$\cot a\sin c=\cot \mathrm{A}\sin \mathrm{B}+\cos c\cos \mathrm{B},$$

we have by differentiation

$$\frac{da}{dc}=(\sin a)^2(\cot a\cot c+\cos \mathrm{B})$$

$$=\cos a\sin a\cot c+(1-(\cos a)^2)\cos \mathrm{B}=\cos \mathrm{B}+\cos a\cot \mathrm{C}\sin \mathrm{B}$$

$$\frac{da}{d\mathrm{A}}=\left(\frac{\sin a}{\sin \mathrm{A}}\right)^2\frac{\sin \mathrm{B}}{\sin c}=\frac{\sin a\sin \mathrm{B}}{\sin \mathrm{C}}$$

$$\frac{da}{d\mathrm{B}}=-\frac{(\sin a)^2}{\sin c}\cot \mathrm{A}\cos \mathrm{B}+\cot c(\sin a)^2\sin \mathrm{B}=\sin a\cot \mathrm{C};$$

and if the *sides* of the triangle be indefinitely diminished, these expressions become, in the limit,

$$\frac{da}{dc}=\frac{\sin(\mathrm{B}+\mathrm{C})}{\sin \mathrm{C}},\quad \frac{da}{d\mathrm{A}}=0,\quad \frac{da}{d\mathrm{B}}=0,$$

provided the angle C *do not vanish.*

If these results be applied to the triangle of which the sides are v, $v_{\prime}$, $\nu_{\prime}-\nu$, and the opposite angles $\pi-\iota_{\prime}$, ι, I, it is easily seen that the values of $\left(\frac{dv}{d\nu}\right)$, &c. are as follows:

$$\left(\frac{dv}{d\nu}\right)=-\frac{\sin\iota_{\prime}}{\sin \mathrm{I}},\ \left(\frac{dv}{d\nu_{\prime}}\right)=\frac{\sin\iota_{\prime}}{\sin \mathrm{I}},\ \left(\frac{dv_{\prime}}{d\nu}\right)=-\frac{\sin\iota}{\sin \mathrm{I}},\ \left(\frac{dv_{\prime}}{d\nu_{\prime}}\right)=\frac{\sin\iota}{\sin \mathrm{I}}$$

$$\left(\frac{dv}{d\iota}\right)=\left(\frac{dv}{d\iota_{\prime}}\right)=\left(\frac{dv_{\prime}}{d\iota}\right)=\left(\frac{dv_{\prime}}{d\iota_{\prime}}\right)=0,$$

provided I be not $=0$. We have therefore

$$\left.\begin{aligned}\frac{d\Omega}{d\nu}&=-\mathrm{E}\Omega+\frac{1}{\sin \mathrm{I}}(\sin\iota_{\prime}.\mathrm{E}\Omega+\sin\iota.\mathrm{E}_{\prime}\Omega)\\ \frac{d\Omega}{d\iota}&=-\frac{d\Omega}{d\mathrm{I}};\end{aligned}\right\}\quad\ldots\ldots\quad(100.)$$

exactly in the same way we find

$$\left.\begin{aligned}\frac{d\Omega_{\prime}}{d\nu_{\prime}}&=-\mathrm{E}_{\prime}\Omega_{\prime}-\frac{1}{\sin \mathrm{I}}(\sin\iota.\mathrm{E}_{\prime}\Omega_{\prime}+\sin\iota_{\prime}.\mathrm{E}\Omega_{\prime})\\ \frac{d\Omega_{\prime}}{d\iota_{\prime}}&=\frac{d\Omega_{\prime}}{d\mathrm{I}}.\end{aligned}\right\}\quad\ldots\ldots\quad(101.)$$

The variations of the elements will now be found by introducing the above values of

$$\frac{d\Omega}{d\nu},\quad \frac{d\Omega_{,}}{d\nu_{,}},\quad \frac{d\Omega}{d\iota},\quad \frac{d\Omega_{,}}{d\iota_{,}}$$

in the expressions given in art. 55, and completing these expressions by the addition of the terms (95.), art. 79. But the angular velocities ω_0, ω_1, ω_2 are no longer wholly arbitrary, since we have made *one* assumption concerning the motion of the axes, which implies *one* relation between these quantities and the elements. In order to determine ω_0, ω_1, ω_2 completely, it will be necessary to make two more assumptions; but first we will investigate the relation already implied.

86. The complete expressions for ν', $\nu'_{,}$, obtained in the way mentioned in the last article, from arts. 55 and 79, may be written in the following form: put for brevity

$$m\sqrt{\mu a(1-e^2)}=p,\quad m_{,}\sqrt{\mu_{,}a_{,}(1-e_{,}^2)}=p_{,},$$

and put $\mu^{\frac{1}{2}}a^{-\frac{3}{2}}$ for n, and $\mu_{,}^{\frac{1}{2}}a_{,}^{-\frac{3}{2}}$ for $n_{,}$; then

$$\left.\begin{aligned}\nu'&=\frac{1}{p\sin\iota}\frac{d\Omega}{d\iota}+\cot\iota.(\omega_0\sin\nu-\omega_1\cos\nu)-\omega_2\\ \nu'_{,}&=\frac{1}{p_{,}\sin\iota_{,}}\frac{d\Omega_{,}}{d\iota_{,}}+\cot\iota_{,}.(\omega_0\sin\nu_{,}-\omega_1\cos\nu_{,})-\omega_2\end{aligned}\right\}\quad\ldots\ldots\quad(102.)$$

and if the latter equation be subtracted from the former, and the conditions

$$\nu_{,}=\nu,\quad \frac{d\Omega}{d\iota}=-\frac{d\Omega}{d\mathrm{I}},\quad \frac{d\Omega_{,}}{d\iota_{,}}=\frac{d\Omega_{,}}{d\mathrm{I}},\quad \iota_{,}-\iota=\mathrm{I}$$

be introduced, the result is easily found to be

$$(\omega_0\sin\nu-\omega_1\cos\nu)\sin\mathrm{I}=\frac{\sin\iota}{p_{,}}\frac{d\Omega_{,}}{d\mathrm{I}}+\frac{\sin\iota_{,}}{p}\frac{d\Omega}{d\mathrm{I}}.\quad\ldots\ldots\quad(103.)$$

This is the relation between ω_0, ω_1 and the elements and t, implied by the one assumed condition that the plane of xy passes through the line of nodes. The angular velocity ω_2 of the system of axes about the axis of z, is so far left, as it evidently ought to be, perfectly arbitrary.

Ω and $\Omega_{,}$ are now functions of the following elements* *only*:—

$$a,\ e,\ (\varepsilon),\ \varpi,\ a_{,},\ e_{,},\ (\varepsilon_{,}),\ \varpi_{,},\ \mathrm{I},\ \nu.$$

And we now have

$$\cos\chi=\cos(\theta-\nu)\cos(\theta_{,}-\nu)+\sin(\theta-\nu)\sin(\theta_{,}-\nu)\cos\mathrm{I}.$$

87. The complete expression for ι', derived from arts. 55 and 79, is easily put in the form

$$\iota'=-\frac{1}{p\sin\iota}\left\{\frac{d\Omega}{d\nu}+(1-\cos\iota)\mathrm{E}\Omega\right\}-(\omega_0\cos\nu+\omega_1\sin\nu);$$

and, on introducing the assumptions of the preceding articles, this will be found to become, after simple reductions,

$$p\sin\mathrm{I}.\iota'=-(\cos\mathrm{I}.\mathrm{E}\Omega+\mathrm{E}_{,}\Omega)-(\omega_0\cos\nu+\omega_1\sin\nu)p\sin\mathrm{I};$$

* On the difference between (ε) and ε see above, art. 55. We cannot strictly call Ω a function of a and ε.

similarly, we find

$$p_{\prime}\sin I . \iota_{\prime}' = \cos I . E_{\prime}\Omega_{\prime} + E\Omega_{\prime} - (\omega_0 \cos\nu + \omega_1 \sin\nu)\, p_{\prime} \sin I;$$

and since $I' = \iota_{\prime}' - \iota'$, we obtain from these

$$pp_{\prime}\sin I . I' = p(\cos I . E_{\prime}\Omega_{\prime} + E\Omega_{\prime}) + p_{\prime}(\cos I . E\Omega + E_{\prime}\Omega), \quad . \; . \; . \quad (104.)$$

an equation which may be transformed as follows:—since

$$\frac{\partial a}{dt} = 0, \quad \frac{\partial e}{dt} = 0 \text{ (art. 79)},$$

we find from art. 55,

$$p' = \frac{d\Omega}{d\varepsilon} + \frac{d\Omega}{d\varpi} = E\Omega,$$

and similarly $p_{\prime}' = E_{\prime}\Omega_{\prime}$, whence it is easily seen that the above equation may be written in the form

$$(pp_{\prime}\cos I)' + pE\Omega_{\prime} + p_{\prime}E_{\prime}\Omega = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (105.)$$

If we investigate, from the above expressions for ι' and $\iota_{\prime}'$, the values of $(p \sin\iota)'$ and $(p_{\prime}\sin\iota_{\prime})'$, we find

$$\sin I . (p \sin\iota)' = -(\cos\iota_{\prime} . E + \cos\iota . E_{\prime})\Omega - p \cos\iota \sin I (\omega_0 \cos\nu + \omega_1 \sin\nu),$$

$$\sin I . (p_{\prime}\sin\iota_{\prime})' = (\cos\iota . E_{\prime} + \cos\iota_{\prime} . E)\Omega_{\prime} - p_{\prime}\cos\iota_{\prime} \sin I (\omega_0 \cos\nu + \omega_1 \sin\nu),$$

and, adding these equations,

$$\sin I . (p \sin\iota + p_{\prime}\sin\iota_{\prime})' = (\cos\iota_{\prime} . E + \cos\iota . E_{\prime})(\Omega_{\prime} - \Omega)$$
$$- (p \cos\iota + p_{\prime}\cos\iota_{\prime}) \sin I . (\omega_0 \cos\nu + \omega_1 \sin\nu). \quad . \; . \; . \quad (106.)$$

With respect to this equation and (103.), it may be observed that $\omega_0 \cos\nu + \omega_1 \sin\nu$ is evidently the *angular velocity of the plane of xy about the line of nodes*, and $\omega_1 \cos\nu - \omega_0 \sin\nu$ is the angular velocity of the same plane about a line in itself perpendicular to the line of nodes; or, which comes to the same thing, the *angular velocity of the line of nodes itself in fixed space, estimated perpendicularly to the plane of xy*.

88. The position of the plane of *xy* at any instant has been so far left arbitrary, except that it has been subjected to the condition of *passing through the line of nodes*.

As a *further assumption*, that which most naturally presents itself is, that *the plane of xy should always coincide with the principal plane*. By the principal plane I mean, of course, that on which at any instant the sum of the projections of the areal velocities (multiplied by the masses) of the two planets about the sun, is a maximum; it evidently always passes through the line of nodes, and would be the invariable plane if the disturbing functions vanished.

To determine the position of this plane we have (see art. 80.) to express the condition that ι and $\iota_{\prime}$ are always so taken, subject to the equation $\iota_{\prime} - \iota = I$, that the expression $m\sqrt{\mu a(1-e^2)} . \cos\iota + m_{\prime}\sqrt{\mu_{\prime}a_{\prime}(1-e_{\prime}^2)} . \cos\iota_{\prime}$ shall be a maximum. We will put σ for the value of this expression, so that using p and $p_{\prime}$ with the same meaning as before (see art. 86.), we have

$$\sigma = p \cos\iota + p_{\prime}\cos\iota_{\prime},$$

and the required condition of a maximum will evidently be

$$p\sin\iota+p_{\prime}\sin\iota_{\prime}=0\,*; \qquad (107.)$$

adding the squares of these expressions, we obtain

$$\sigma^2=p^2+p_{\prime}^2+2pp_{\prime}\cos \mathrm{I}, \qquad (108.)$$

which determines the actual value of σ; moreover we have

$$-\frac{\sin\iota}{p_{\prime}}=\frac{\sin\iota_{\prime}}{p}=\frac{\sin \mathrm{I}}{\sigma}; \qquad (109.)$$

and it is easy to find

$$\left.\begin{array}{l}\sigma\cos\iota=p+p_{\prime}\cos \mathrm{I}\\ \sigma\cos\iota_{\prime}=p_{\prime}+p\cos \mathrm{I}\end{array}\right\}, \qquad (110.)$$

so that σ, $\sin\iota$, $\sin\iota_{\prime}$, $\cos\iota$, $\cos\iota_{\prime}$ are all simply expressible in terms of p, $p_{\prime}$ and I. The variation of σ is easily found by means of the equation (105.), which gives

$$\sigma\sigma'=(p_{\prime}\mathrm{E}_{\prime}-p\mathrm{E})(\Omega_{\prime}-\Omega). \qquad (111.)$$

The equation (103.) now gives (see (109.))

$$\sigma(\omega_1\cos\nu-\omega_0\sin\nu)=\frac{d}{d\mathrm{I}}(\Omega_{\prime}-\Omega); \qquad (112.)$$

and from (106.) we obtain

$$\sigma^2\sin \mathrm{I}(\omega_0\cos\nu+\omega_1\sin\nu)=\{(p+p_{\prime}\cos \mathrm{I})\mathrm{E}_{\prime}+(p_{\prime}+p\cos \mathrm{I})\mathrm{E}\}(\Omega_{\prime}-\Omega). \qquad (113.)$$

The two last equations determine the motion of the principal plane in space, irrespectively of any arbitrary *sliding* motion which we may attribute to it in its own plane. For they give the angular velocities with which it is at any instant moving about two lines at right angles to one another in its own plane (see the end of art. 87.). They may be put in another form as follows:—the actual value of $\Omega_{\prime}-\Omega$ is

$$mm_{\prime}\left(\frac{r}{r_{\prime}^2}-\frac{r_{\prime}}{r^2}\right)\cos\chi,$$

where $\cos\chi$ has the value given above (end of art. 86.); and if the operations indicated be actually performed, observing that $\mathrm{E}r=0$, $\mathrm{E}_{\prime}r=0$, &c., and that

$$\mathrm{E}\cos\chi=\frac{d\cos\chi}{d\theta}, \text{ \&c.},$$

the results will be found to be

$$\sigma(\omega_1\cos\nu-\omega_0\sin\nu)=mm_{\prime}\sin \mathrm{I}.\left(\frac{r_{\prime}}{r^2}-\frac{r}{r_{\prime}^2}\right)\sin(\theta-\nu)\sin(\theta_{\prime}-\nu)$$

$$\sigma^2(\omega_0\cos\nu+\omega_1\sin\nu)=mm_{\prime}\sin \mathrm{I}.\left(\frac{r_{\prime}}{r^2}-\frac{r}{r_{\prime}^2}\right)\times(p\cos(\theta-\nu)\sin(\theta_{\prime}-\nu)+p_{\prime}\sin(\theta-\nu)\cos(\theta_{\prime}-\nu)).$$

Here $\theta-\nu$, $\theta_{\prime}-\nu$ represent, it will be remembered, the angular distances of the planets from the line of nodes.

We will assume for the present the condition $\omega_2=0$, so that the plane of xy may have no sliding motion, but *roll* upon the conical surface to which it is always a tan-

* Referring to the arrangement supposed in the diagram, it will be seen that ι becomes *negative* in the case now considered.

gent plane. If, then, we call ω the actual angular velocity of the principal plane about its instantaneous axis of rotation, so that $\omega=\sqrt{\omega_0^2+\omega_1^2}$, and put j for the angle between the line of nodes and the instantaneous axis (which is the line in which the principal plane is intersected by its consecutive), we shall have

$$\omega_0\cos\nu+\omega_1\sin\nu=\omega\cos j \text{ and } \omega_1\cos\nu-\omega_0\sin\nu=\omega\sin j;$$

and the above equations give

$$\sigma\cot j=p\cot(\theta-\nu)+p_{,}\cot(\theta_{,}-\nu),$$

a result which may also be put in the form (see (109.))

$$\sin\mathrm{I}\cot j=\sin\iota_{,}\cot(\theta-\nu)-\sin\iota\cot(\theta_{,}-\nu).$$

It is very easy to show by spherical trigonometry that this equation signifies that the instantaneous axis coincides with the line in which the principal plane is intersected by the plane of the two radii vectores*. In other words, we have this theorem:

The principal plane always turns about the line in which it is intersected by the plane of the two radii vectores.

It follows of course that the principal plane always touches the conical surface described in fixed space (relatively to the sun) by the said line. I think it probable that most persons would expect, at first sight, that the principal plane would always touch the conical surface described by the *line of nodes*, which, as has been just shown, is not the case. It is perhaps worth while to verify this result by independent reasoning.

89. Let Roman letters refer, as in former articles, to axes of coordinates having any arbitrary *fixed* directions. Then, putting $\mathrm{A}=m(\mathrm{yz}'-\mathrm{zy}')$, &c. (as in art. 80.), and using ξ, η, ζ for current coordinates, the equation to the principal plane is

$$(\mathrm{A}+\mathrm{A}_{,})\xi+(\mathrm{B}+\mathrm{B}_{,})\eta+(\mathrm{C}+\mathrm{C}_{,})\zeta=0; \quad \ldots\ldots \quad (114.)$$

and the line in which this plane is cut by its consecutive is determined by combining the equation (114.) with that obtained from it by differentiating the coefficients of ξ, η, ζ with respect to t; namely,

$$(\mathrm{A}'+\mathrm{A}'_{,})\xi+(\mathrm{B}'+\mathrm{B}'_{,})\eta+(\mathrm{C}'+\mathrm{C}'_{,})\zeta=0. \quad \ldots\ldots \quad (115.)$$

Now from the fundamental equations

$$m\mathrm{x}''+m\mu\frac{\mathrm{x}}{\mathrm{r}^3}=\frac{d\Omega}{d\mathrm{x}}, \text{ \&c.,}$$

we obtain $$\mathrm{A}'=m(\mathrm{yz}''-\mathrm{zy}'')=\mathrm{y}\frac{d\Omega}{d\mathrm{z}}-\mathrm{z}\frac{d\Omega}{d\mathrm{y}}=mm_{,}(\mathrm{yz}_{,}-\mathrm{zy}_{,})(\delta^{-3}-\mathrm{r}^{-3})$$

* Let ψ be the angle between the principal plane and the plane of the radii vectores; the former plane divides the angle I into two parts, of which one is $\iota_{,}$ and the other is $-\iota$; and we get two spherical triangles which have a common side j, with adjacent angles $\iota_{,}$ and ψ in one, $-\iota$ and $\pi-\psi$ in the other; hence

$$\cot(\theta-\nu)\sin j=\cot\psi\sin\iota+\cos j\cos\iota$$
$$\cot(\theta_{,}-\nu)\sin j=\cot\psi\sin\iota_{,}+\cos j\cos\iota_{,};$$

and if $\cot\psi$ be eliminated between these, the result is the equation in the text.

(where δ is the distance between m and $m_{,}$); and in like manner

$$A'_{,}=-mm_{,}(yz_{,}-zy_{,})(\delta^{-3}-r_{,}^{-3});$$

we have therefore

$$A'+A'_{,}=mm_{,}(yz_{,}-zy_{,})(r_{,}^{-3}-r^{-3}),$$

with similar expressions for $B'+B'_{,}$, $C'+C'_{,}$; so that, when the common factor $mm_{,}(r_{,}^{-3}-r^{-3})$ is omitted, the equation (115.) becomes

$$(yz_{,}-zy_{,})\xi+(zx_{,}-xz_{,})\eta+(xy_{,}-yx_{,})\zeta=0,$$

which is evidently the equation to the plane containing the two radii vectores. Thus the theorem in question is verified.

90. To return from this digression: the motion of the line of nodes *in the principal plane* will be given by putting $\omega_2=0$ in either of the equations (102.), and introducing the value of $\omega_0\sin\nu-\omega_1\cos\nu$ from (103.). In this way we find, after slight reductions,

$$pp_{,}\sin I.\nu'=p_{,}\cos\iota_{,}\frac{d\Omega}{dI}+p\cos\iota\frac{d\Omega_{,}}{dI},$$

in which we may substitute for $\cos\iota$, $\cos\iota_{,}$, the values given by equations (110.); this gives

$$\sigma pp_{,}\sin I.\nu'=(p_{,}^2+pp_{,}\cos I)\frac{d\Omega}{dI}+(p^2+pp_{,}\cos I)\frac{d\Omega_{,}}{dI};$$

or, if we introduce the actual values of $\frac{d\Omega}{dI}$, $\frac{d\Omega_{,}}{dI}$, we find

$$pp_{,}\nu'=-mm_{,}rr_{,}\sin(\theta-\nu)\sin(\theta_{,}-\nu)\times\{p\cos\iota.(\delta^{-3}-r^{-3})+p_{,}\cos\iota_{,}(\delta^{-3}-r_{,}^{-3})\}.$$

It is not my purpose however to enter further into details; and I shall conclude this subject by briefly examining the consequences of a slightly different assumption as to the motion of the axes of coordinates. I shall suppose, namely, that the plane of xy still always coincides with the principal plane, but has a *sliding* motion such that the axis of x *always coincides with the line of nodes.*

91. The assumption made at the end of the last article implies the condition $\nu=0$; and ω_2 will no longer be 0, but must be determined by equations (102.); either of these gives (putting $\nu'=\nu'_{,}=0$, and reducing by means of (103.), (109.), &c.)

$$pp_{,}\sin I.\omega_2=p_{,}\cos\iota_{,}\frac{d\Omega}{dI}+p\cos\iota\frac{d\Omega_{,}}{dI}, \quad \ldots\ldots \quad (116.)$$

which coincides, as of course it ought, with the expression given for ν' on the former hypothesis (art. 90.). The difference is that Ω, $\Omega_{,}$ now no longer contain ν.

The values of ω_0, ω_1 are obtained at once by putting $\nu=0$ in the equations (112.), (113.); and all the conclusions which were derived independently of any supposition as to the value of ω_2, subsist as before, when modified by putting $\nu=0$.

We may add one more equation, which is required in forming some of the expressions for the variations of the elements; namely,

$$\tan\frac{\iota}{2}=\frac{-p_{,}\sin I}{\sigma+p+p_{,}\cos I}. \quad \ldots\ldots \quad (117.)$$

This is easily obtained from (109.) and (110.); and in like manner

$$\tan\frac{\iota_{,}}{2}=\frac{p\sin \mathrm{I}}{\sigma+p_{,}+p\cos \mathrm{I}}.$$

I shall now recapitulate the results of the last supposition, so as to exhibit in one view the transformed differential equations of the problem of three bodies. It will be as well to repeat also the explanation of the symbols.

92. *Signification of the Symbols.*

M, m, $m_{,}$ are the masses of the sun, and of the two planets, and $\mathrm{M}+m=\mu$, $\mathrm{M}+m_{,}=\mu_{,}$.

a and e are the semiaxis and excentricity of the instantaneous ellipse described by m about the sun.

$a_{,}$ and $e_{,}$ have the same meanings with reference to $m_{,}$.

I is the inclination of the plane of the former ellipse to that of the latter.

θ, $\theta_{,}$ are the longitudes of the two planets, measured in the planes of their orbits *from the common line of nodes.*

r, $r_{,}$ their radii vectores.

ϖ, $\varpi_{,}$ the longitudes of the perihelia, measured likewise in the planes of the orbits *from the line of nodes.*

ε, $\varepsilon_{,}$ two elements such that $\int_0^t ndt+\varepsilon$, $\int_0^t n_{,}dt+\varepsilon_{,}$ are the *mean longitudes*, where n, $n_{,}$ are defined as usual by the equations $n^2=\mu a^{-3}$, $n_{,}^2=\mu_{,}a_{,}^{-3}$.

Then $\int_0^t ndt+\varepsilon-\varpi$ is the mean anomaly of m, and r, θ are functions of the mean anomaly and mean longitude given by the laws of elliptic motion. The same is true for $m_{,}$, *mutatis mutandis.*

χ is the angle between the radii vectores, so that

$$\cos\chi=\cos\theta\cos\theta_{,}+\sin\theta\sin\theta_{,}\cos \mathrm{I}.$$

Let δ be the distance between the planets, so that

$$\delta^2=r^2+r_{,}^2-2rr_{,}\cos\chi.$$

Ω, $\Omega_{,}$ are the disturbing functions, defined by the equations

$$\Omega=mm_{,}\left(\frac{1}{\delta}-\frac{r\cos\chi}{r_{,}^2}\right),\quad \Omega_{,}=mm_{,}\left(\frac{1}{\delta}-\frac{r_{,}\cos\chi}{r^2}\right),$$

and, when expressed in terms of the elements and t, are functions of

$$a,\ a_{,},\ e,\ e_{,},\ \mathrm{I},\ \int_0^t ndt+\varepsilon,\ \int_0^t ndt+\varepsilon-\varpi,\ \int_0^t n_{,}dt+\varepsilon_{,},\ \int_0^t n_{,}dt+\varepsilon_{,}-\varpi_{,}.$$

When Ω, $\Omega_{,}$ are considered on the one hand as expressed in this way, and on the other, in their original form as functions of r, $r_{,}$, θ, $\theta_{,}$, and I, we have, as applied to either of them,

$$\frac{d}{d\theta}=\frac{d}{d\varepsilon}+\frac{d}{d\varpi},\quad \frac{d}{d\theta_{,}}=\frac{d}{d\varepsilon_{,}}+\frac{d}{d\varpi_{,}}.$$

$p, p_{\prime}, \sigma$ are defined by the equations

$$p=m\sqrt{\mu a(1-e^2)},\quad p_{\prime}=m_{\prime}\sqrt{\mu_{\prime}a_{\prime}(1-e_{\prime}^2)},\quad \sigma^2=p^2+p_{\prime}^2+2pp_{\prime}\cos \mathrm{I}.$$

$\iota, \iota_{\prime}$ are the angles between the *principal plane* and the planes of the two orbits, and are given functions of the elements $a, a_{\prime}, e, e_{\prime}$, I, by virtue of the equations

$$-p\sigma\sin\iota=p_{\prime}\sigma\sin\iota_{\prime}=pp_{\prime}\sin \mathrm{I},$$

from which we have also

$$\sigma\cos\iota=p+p_{\prime}\cos\mathrm{I},\quad \sigma\cos\iota_{\prime}=p_{\prime}+p\cos\mathrm{I}$$

(for the values of $\tan\frac{\iota}{2}$, $\tan\frac{\iota_{\prime}}{2}$, see art. 91.).

ω_0 is the angular velocity of the principal plane about the line of nodes;

ω_1 the angular velocity of the principal plane about a line in itself perpendicular to the line of nodes;

ω_2 the angular velocity of the line of nodes estimated in the direction of the principal plane.

Differential Equations of the Problem.

93. The nine *intrinsic elements*, as we may perhaps appropriately call them, namely,

$$a,\ a_{\prime},\ e,\ e_{\prime},\ \varepsilon,\ \varepsilon_{\prime},\ \varpi,\ \varpi_{\prime},\ \mathrm{I},$$

are determined as functions of t by the following system of nine simultaneous differential equations of the first order:

$$m\mu a'=2na^2\frac{d\Omega}{d\varepsilon},\quad m_{\prime}\mu_{\prime}a_{\prime}'=2n_{\prime}a_{\prime}^2\frac{d\Omega_{\prime}}{d\varepsilon_{\prime}}$$

$$m\mu e'=-\frac{na\sqrt{1-e^2}}{e}\left\{\frac{d\Omega}{d\varpi}+(1-\sqrt{1-e^2})\frac{d\Omega}{d\varepsilon}\right\}$$

$$m_{\prime}\mu_{\prime}e_{\prime}'=-\frac{n_{\prime}a_{\prime}\sqrt{1-e_{\prime}^2}}{e_{\prime}}\left\{\frac{d\Omega_{\prime}}{d\varpi_{\prime}}+(1-\sqrt{1-e_{\prime}^2})\frac{d\Omega_{\prime}}{d\varepsilon_{\prime}}\right\}$$

$$m\mu\varepsilon'=-2na^2\frac{d\Omega}{da}+\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{d\Omega}{de}$$
$$-\frac{m\mu}{\sin\mathrm{I}}\left\{\frac{\cos\mathrm{I}}{p}\frac{d\Omega}{d\mathrm{I}}+\frac{1}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}\right\}$$

$$m_{\prime}\mu_{\prime}\varepsilon_{\prime}'=-2n_{\prime}a_{\prime}^2\frac{d\Omega_{\prime}}{da_{\prime}}+\frac{n_{\prime}a_{\prime}\sqrt{1-e_{\prime}^2}}{e_{\prime}}(1-\sqrt{1-e_{\prime}^2})\frac{d\Omega_{\prime}}{de_{\prime}}$$
$$-\frac{m_{\prime}\mu_{\prime}}{\sin\mathrm{I}}\left\{\frac{\cos\mathrm{I}}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}+\frac{1}{p}\frac{d\Omega}{d\mathrm{I}}\right\}$$

$$m\mu\varpi'=\frac{na\sqrt{1-e^2}}{e}\frac{d\Omega}{de}-\frac{m\mu}{\sin\mathrm{I}}\left\{\frac{\cos\mathrm{I}}{p}\frac{d\Omega}{d\mathrm{I}}+\frac{1}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}\right\}$$

$$m_{\prime}\mu_{\prime}\varpi_{\prime}'=\frac{n_{\prime}a_{\prime}\sqrt{1-e_{\prime}^2}}{e_{\prime}}\frac{d\Omega_{\prime}}{de_{\prime}}-\frac{m_{\prime}\mu_{\prime}}{\sin\mathrm{I}}\left\{\frac{\cos\mathrm{I}}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}+\frac{1}{p}\frac{d\Omega}{d\mathrm{I}}\right\}$$

$$pp_{\prime} \sin \mathrm{I}.\mathrm{I}'=p_{\prime}\left\{\cos \mathrm{I}\left(\frac{d\Omega}{d\varepsilon}+\frac{d\Omega}{d\varpi}\right)+\frac{d\Omega}{d\varepsilon_{\prime}}+\frac{d\Omega}{d\varpi_{\prime}}\right\}$$

$$+p\left\{\cos \mathrm{I}\left(\frac{d\Omega_{\prime}}{d\varepsilon_{\prime}}+\frac{d\Omega_{\prime}}{d\varpi_{\prime}}\right)+\frac{d\Omega_{\prime}}{d\varepsilon}+\frac{d\Omega_{\prime}}{d\varpi}\right\}.$$

94. The only parts of the preceding expressions of which the deduction is not perfectly obvious, are the terms involving I in the values of ε', $\varepsilon_{\prime}'$, ϖ', $\varpi_{\prime}'$. They are obtained, as has been sufficiently explained, from the expressions in art. 55, to which are to be added the values of $\frac{\partial\varepsilon}{dt}$, &c. ((95.), art. 79.); on putting $\nu=0$, ω_0 disappears from the latter; and the values of ω_1, ω_2, $\cos\iota$, $\cos\iota_{\prime}$, $\tan\frac{\iota}{2}$, $\tan\frac{\iota_{\prime}}{2}$ are to be introduced (equations (110.), (112.), (116.), (117.)). After some rather troublesome reductions, the expressions above given will be found.

In these equations it will be recollected that the mean longitude of m is represented by $\int_0^t ndt+\varepsilon$, and the differentiation with respect to a in $\frac{d\Omega}{da}$ is only performed so far as a appears explicitly. If we wished that the mean longitude should be expressed by $nt+\varepsilon$, the only change in the equations would be that the differentiation with respect to a must be total; i.e. must extend to a as contained in n. A similar remark applies of course to $\varepsilon_{\prime}$.

In actual use it would be more convenient to introduce R, $\mathrm{R}_{\prime}$ instead of Ω, $\Omega_{\prime}$; the latter functions give a rather more symmetrical form to the equations, and are more convenient in general investigations. (The relation between them here is merely $\Omega=m\mathrm{R}$, $\Omega_{\prime}=m_{\prime}\mathrm{R}_{\prime}$; in another part of the paper the symbol Ω was used for $-m\mathrm{R}$ (art. 55.)).

95. If the equations of art. 93. were completely integrated, the *intrinsic* motion of the system would be completely determined; that is, we should know at any instant the dimensions of the two orbits, the mutual inclination of their planes, the position of their axes with respect to the line of nodes, the place of each planet in its orbit, and (by (110.)) the inclination of each orbit to the principal plane.

The position of the system relatively to fixed space would then have to be separately determined as follows:—

The three quantities ω_0, ω_1, ω_2 (see end of art. 92.), of which the values are ((112.), (113.), (116.)) given by the equations

$$\sigma^2 \sin \mathrm{I}.\omega_0=\left\{(p+p_{\prime}\cos \mathrm{I})\left(\frac{d}{d\varepsilon_{\prime}}+\frac{d}{d\varpi_{\prime}}\right)+(p_{\prime}+p\cos \mathrm{I})\left(\frac{d}{d\varepsilon}+\frac{d}{d\varpi}\right)\right\}(\Omega_{\prime}-\Omega)$$

$$\sigma\omega_1=\frac{d}{d\mathrm{I}}(\Omega_{\prime}-\Omega)$$

$$\sigma \sin \mathrm{I}.\omega_2=\left(\frac{p_{\prime}}{p}+\cos \mathrm{I}\right)\frac{d\Omega}{d\mathrm{I}}+\left(\frac{p}{p_{\prime}}+\cos \mathrm{I}\right)\frac{d\Omega_{\prime}}{d\mathrm{I}},$$

would be given functions of t. Then if we call

J the inclination of the principal plane to an arbitrary fixed plane;

Ω the longitude of the line of intersection of these two planes, reckoned in the fixed plane from a fixed line;

N the angle between this line of intersection and the *line of nodes*;

we should have (as in art. 83. with a different notation)

$$\left.\begin{aligned} J'&=\omega_0\cos N-\omega_1\sin N\\ \Omega'\sin J&=\omega_0\sin N+\omega_1\cos N\\ N'&=\omega_2-\cot J(\omega_0\sin N+\omega_1\cos N)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (118.)$$

and the integration of this system would give J, Ω, N as functions of t, and so determine at any instant the position of the principal plane and of the line of nodes, relatively to fixed space.

With respect to the motion of the principal plane, the following may be added. It has already been shown (art. 88, 89.) that the line about which it is at any instant turning, coincides with that in which it is intersected by the plane of the radii vectores; and the values of ω_0, ω_1 (see art. 88, putting $\nu=0$ in the expressions there given) may be put in the form

$$\sigma^2\omega_0=mm_{\prime}\sin I.\left(\frac{r_{\prime}}{r^2}-\frac{r}{r_{\prime}^2}\right)(p\cos\theta\sin\theta_{\prime}+p_{\prime}\sin\theta\cos\theta_{\prime})$$

$$\sigma\omega_1=mm_{\prime}\sin I\left(\frac{r_{\prime}}{r^2}-\frac{r}{r_{\prime}^2}\right)\sin\theta\sin\theta_{\prime}.$$

If the latter of these be multiplied by σ, and then both sides of each squared, and the results added (after putting for σ^2 on the right its value $p^2+p_{\prime}^2+2pp_{\prime}\cos I$), we find, observing that $\cos\theta\cos\theta_{\prime}+\sin\theta\sin\theta_{\prime}\cos I=\cos\chi$,

$$\sigma^2\sqrt{\omega_0^2+\omega_1^2}=mm_{\prime}\sin I\left(\frac{r_{\prime}}{r^2}-\frac{r}{r_{\prime}^2}\right)(p^2\sin^2\theta_{\prime}+p_{\prime}^2\sin^2\theta+2pp_{\prime}\sin\theta\sin\theta_{\prime}\cos\chi)^{\frac{1}{2}},$$

an expression which may be further transformed as follows. Let λ, $\lambda_{\prime}$ be the *latitudes* of m, $m_{\prime}$ (with reference to the principal plane); then $\sin\lambda=\sin\theta\sin\iota$, $\sin\lambda_{\prime}=\sin\theta_{\prime}\sin\iota_{\prime}$; hence, since $p=\frac{\sigma\sin\iota_{\prime}}{\sin I}$, $p_{\prime}=\frac{-\sigma\sin\iota}{\sin I}$, we obtain

$$\sigma\sqrt{\omega_0^2+\omega_1^2}=mm_{\prime}\left(\frac{r_{\prime}}{r^2}-\frac{r}{r_{\prime}^2}\right)(\sin^2\lambda+\sin^2\lambda_{\prime}-2\sin\lambda\sin\lambda_{\prime}\cos\chi)^{\frac{1}{2}}.$$

This gives the absolute angular velocity with which the principal plane is at any instant changing its direction in space; it is evident that (if the supposition $r_{\prime}=r$ be excluded) it can never vanish except when both planets are in the line of nodes. The *direction* of the rotation is determined by the signs of ω_0 and ω_1.

96. The system of differential equations given in art. 93. affords an example of the so-called "elimination of the nodes" in the problem of three bodies. JACOBI, by a very remarkable and ingenious transformation, has effected the elimination in a quite

different manner*. The equations of art. 93. are *merely* transformations of the original differential equations of the problem, without any integrations; they are however in a form which might perhaps be used advantageously in certain cases for the purposes of physical astronomy. Those of JACOBI are obtained by employing all the four usual integrals of the problem, and are shown to include an *additional integration.* They have however the disadvantage of substituting the coordinates of two fictitious bodies for those of the actual planets, and would probably be inconvenient for ordinary practical use; though in a theoretical point of view they seem to deserve more attention than they have hitherto received. It would be wrong to take leave of this celebrated problem without referring to another transformation by M. BERTRAND†, which, as has been remarked by a recent writer in the same journal, effects *six* integrations, and therefore represents the furthest advance which has yet been made towards a rigorous solution.

APPENDIX A.

When the method described in Theorem VII. (art. 49.) is applied to the solution of a system of equations of the form (I.), of which n integrals, $a_1 \dots a_n$, satisfying the conditions $[a_i, a_j]=0$, are given, the first step is to express the $n+1$ partial differential coefficients $\frac{dX}{dx_1}$, &c., and $\frac{dX}{dt}$; namely, the values of $y_1, \dots y_n$ and $-Z$ in terms of $x_1, \dots x_n, a_1, \dots a_n$ and t. The *direct* process is then to find X by integrating the expression $y_1dx_1+y_2dx_2+\dots+y_ndx_n-Zdt$, and afterwards to form the remaining integral equations $\frac{dX}{da_1}=b_1$, &c.: when this process is adopted, the inferior limits in the integrations are perfectly arbitrary; in other words, we may add to X an arbitrary function of $a_1, a_2, \dots a_n$, without altering any of the general properties of the final system of integrals.

But it is generally much more convenient to perform the differentiations with respect to $a_1, \dots a_n$ *first*, and integrate afterwards; thus we obtain the remaining equations in the form

$$b_i=\int\left(\frac{dy_1}{da_i}dx_1+\frac{dy_2}{da_i}dx_2+\dots+\frac{dy_n}{da_i}dx_n-\frac{dZ}{da_i}dt\right).$$

When this plan is followed, the limits are still arbitrary if it be only required that the equations thus obtained shall be *true*; but if it be required that they shall form, with the given integrals, a *normal solution*, it is necessary to take the limits in such a manner that the functions equated to $b_1, b_2, \dots b_n$ shall be the partial differential coefficients with respect to $a_1, a_2, \dots a_n$, of *one and the same function*; which will not generally be the case unless care be taken that it should be so.

In practice, the expression for dX usually consists of several terms, of which each

* Comptes Rendus, 1842, part 2. p. 236, &c.

† LIOUVILLE's Journal, 1852.

contains one of the variables *only*. Suppose one of these terms is

$$\phi(x, a_1, a_2, \dots a_n)dx,$$

so that, so far as this term is concerned, we have

$$X=\int_A^x \phi(x, a_1, a_2, \dots a_n)dx,$$

where A is an arbitrary function of $a_1, \dots a_n$. Consequently

$$\frac{dX}{da_i}=\int_A^x \frac{d\phi}{da_i}dx-\phi(A, a_1, \dots a_n)\frac{dA}{da_i},$$

and we see that we should not *in general* obtain the differential coefficients with respect to $a_1, \dots a_n$ of one and the same function X, by merely integrating $\frac{d\phi}{da_1}$, $\frac{d\phi}{da_2}$, &c., with respect to x, from the same inferior limit A, chosen at hazard.

But it is evident that we shall attain this end if we adopt the following simple rule:—

Integrate $\frac{d\phi}{da_1}$, &c. with respect to x, taking the same inferior limit in each case, namely, either

(1) a value A of x which satisfies the equation $\phi(x, a_1, \dots a_n)=0$, or

(2) any *determinate* constant (i. e. not a function of $a_1, \dots a_n$).

For example, in the problem of central forces (Part I., art. 28, &c.), we had (see art. 19.)

$$dX=-hdt+cd\theta+\left(2m(h+\phi(r))-k^2r^{-2}\right)^{\frac{1}{2}}dr+(k^2-c^2\sec^2\lambda)^{\frac{1}{2}}d\lambda$$

(where r, θ, λ are the three variables).

The very troublesome process of differentiating X with respect to h, k and c *after* first finding X by integrating the above expression, is avoided by the method adopted in art. 29; namely, by differentiating first, and integrating afterwards. In the integrations with respect to r, the inferior limit is one of the roots of the equation

$$2m(h+\phi(r))-k^2r^{-2}=0,$$

namely (in the case of elliptic motion), the perihelion distance; and in those with respect to λ, the inferior limit is 0; so that the rule above given is observed.

At the time of writing the article referred to, neither the rule itself, nor the necessity of attending to the limits, had occurred to me; it was therefore, strictly speaking, accidental that the final integrals were obtained in a *normal form*.

In treating the problem of rotation (Section III.), I perceived the necessity of caution as to the limits, if the former order of proceeding were adopted; but preferred avoiding the risk of error altogether, by performing the integrations first, so as to obtain the actual expression for V. The final equations (R.), art. 44, might however be obtained in a more simple way by differentiating *first*; thus we should have (see equations (45.), (46.)), observing that $\frac{dk}{dh}=\frac{k}{2h}$, &c. (art. 44.), and putting

$$(1-\cos^2 i-\cos^2 j+2\cos i\cos j\cos\theta-\cos^2\theta)^{\frac{1}{2}}=\mathrm{Q},$$

$$\frac{d\mathrm{V}}{dh}=\frac{k}{2h}\left\{\psi\cos i+\phi\cos j+\int\frac{\mathrm{Q}}{\sin\theta}d\theta\right\}=t+\tau$$

$$\frac{d\mathrm{V}}{d\cos i}=k\left\{\psi+\int\frac{\cos j\cos\theta-\cos i}{\mathrm{Q}\sin\theta}d\theta\right\}=\alpha \quad \text{(i.)}$$

$$\frac{d\mathrm{V}}{d\cos j}=\frac{(\mathrm{C}-\mathrm{A})k^3\cos j}{2\mathrm{AC}h}\left\{\psi\cos i+\phi\cos j+\int\frac{\mathrm{Q}}{\sin\theta}d\theta\right\}$$

$$+k\left\{\phi+\int\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta\right\}=\beta.$$

In order to get rid of the troublesome integration involved in the term $\int\frac{\mathrm{Q}}{\sin\theta}d\theta$, we may (1) eliminate this term between the first and last of these equations, and (2) eliminate ψ between the first and second. We thus find the two following equations,

$$\phi+\int\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta=\frac{\beta}{k}-\frac{\mathrm{C}-\mathrm{A}}{\mathrm{AC}}k\cos j.(t+\tau) \quad \text{(ii.)}$$

$$\cos j\left\{\phi+\int\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta\right\}+\int\frac{\sin\theta d\theta}{\mathrm{Q}}=\frac{2h}{k}(t+\tau)-\frac{\alpha\cos i}{k},$$

which last, combined with the preceding, gives

$$\int\frac{\sin\theta d\theta}{\mathrm{Q}}=\frac{k}{\mathrm{A}}(t+\tau)-\frac{\alpha\cos i+\beta\cos j}{k}; \quad \text{(iii.)}$$

and we may take (i.), (ii.) and (iii.) as expressing the solution of the problem.

Now we have $\cos\mathrm{I}=\frac{\cos i-\cos j\cos\theta}{\sin j\sin\theta}$, from which it is easy to find (observing the conditions which determine the sign of Q)

$$-d\mathrm{I}=\frac{\cos i\cos\theta-\cos j}{\mathrm{Q}\sin\theta}d\theta, \text{ and similarly,}$$

$$-d\mathrm{J}=\frac{\cos j\cos\theta-\cos i}{\mathrm{Q}\sin\theta}d\theta;$$

we have moreover $\quad d\Theta=\frac{\sin\theta d\theta}{\mathrm{Q}}.$

All the integrations may therefore now be performed immediately; and we may take for the inferior limit of θ any value which satisfies the equation $\mathrm{Q}=0$, or

$$(\cos\theta-\cos i\cos j)^2-\sin^2 i\sin^2 j=0;$$

this is satisfied by $\theta=i+j$, which evidently corresponds to $\mathrm{I}=0$, $\mathrm{J}=0$, $\Theta=0$, and it is manifest that equations (i.), (ii.), (iii.) will thus become identical with equations (R) of art. 44.

I do not regret however having introduced the rather prolix investigation of arts. 39 and 43, because it is interesting to know the actual value of V (equation (48.)), which the method just given leaves undetermined.

APPENDIX B.

On the subject of the transformation of elements, the following additional remarks will hardly be superfluous. Suppose Ω is originally a function of the elements a, b, c, &c. with t; and let α, β, γ, &c. be other quantities connected with a, b, c, &c., by equations such as

$$da = \mathrm{A}d\alpha + \mathrm{B}d\beta + \mathrm{C}d\gamma + \ldots + \mathrm{K}dt, \quad \ldots \ldots \quad (a.)$$

where A, B, C, ... K are given functions of α, β, γ, ... t; or by equations such as

$$d\alpha = \mathrm{A}_{,}da + \mathrm{B}_{,}db + \ldots + \mathrm{K}_{,}dt, \quad \ldots \ldots \quad (\alpha.)$$

where $\mathrm{A}_{,}$, &c. are given functions of a, b, ..., t. In either case, if each of the equations be integrable *per se*, we may consider a, b, c, ... as functions of α, β, γ, ..., t; and such equations as

$$\frac{d\Omega}{d\alpha} = \mathrm{A}\frac{d\Omega}{da} + \mathrm{B}\frac{d\Omega}{db} + \ldots \quad \ldots \ldots \quad (\Omega.)$$

are both significant and true.

But if the expressions on the right of the equations (a.) be *not* differentials *per se*, the equations (Ω.) are either unmeaning or untrue. For the symbol $\frac{d\Omega}{d\alpha}$ implies one of two things; either, that Ω is expressed in terms of α, β, ... t *without* arbitrary constants (i. e. that the transformation of Ω can be actually effected *without* integrating the differential equations of the problem), which is manifestly impossible, unless (a.), &c. be integrable *per se*; or else, that the differential equations are to be conceived to have been completely integrated, so that a, b, &c., and consequently $\mathrm{A}_{,}$, $\mathrm{B}_{,}$, &c., are known as *functions of* t *and arbitrary constants*, whereby the right-hand side of (α.) becomes an *explicit function of* t (and arbitrary constants), so that α, β, &c. may by integration be expressed in the same way, and, by means of (a.), a, b, &c. may be similarly expressed, and finally, by algebraical elimination, a, b, &c. become functions of α, β, &c., t, and arbitrary constants. On this supposition, $\frac{d\Omega}{d\alpha}$ has a meaning, but the equation (Ω.) is *untrue*; for we must have

$$\frac{d\Omega}{d\alpha} = \frac{d\Omega}{da}\frac{da}{d\alpha} + \frac{d\Omega}{db}\frac{db}{d\alpha} + \ldots;$$

and it is manifestly not true that $\frac{da}{d\alpha} = \mathrm{A}$, &c. in this case, because the equation

$$da = \mathrm{A}d\alpha + \mathrm{B}d\beta + \ldots + \mathrm{K}dt,$$

not being integrable *per se*, only subsists for those variations of α, β, &c. which *actually take place* during the instant dt; whereas the equation

$$da = \frac{da}{d\alpha}d\alpha + \frac{da}{d\beta}d\beta + \ldots + \frac{da}{dt}dt$$

subsists for *arbitrary variations* of all the variables. This view of the subject entirely coincides, in substance, with that taken by JACOBI; but the above mode of stating it may tend to make it clearer, and to call attention to a matter which, so far as I know, is not so much as mentioned in any of the elementary works usually in the hands of students of physical astronomy.

[*Addition to* APPENDIX B.]

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The remark made above, that the symbol $\frac{d\Omega}{d\alpha}$ is *unmeaning* in the case considered, is not of course intended to imply that a meaning *may not be given to it*; but then such meaning is different from the ordinary signification of the symbol, which is a *partial derived function.*

The whole matter may be strikingly illustrated by a simple example.

Consider the movement of a rigid body about a fixed point. Adopting the notation of art. 40 (Part I.), we have

$$pdt = -\cos\phi d\theta - \sin\theta\sin\phi d\psi$$
$$qdt = \sin\phi d\theta - \sin\theta\cos\phi d\psi$$
$$rdt = d\phi + \cos\theta d\psi.$$

Let α, β, γ be three new variables, defined by the equations

$$\alpha = \int_0^t pdt, \quad \beta = \int_0^t qdt, \quad \gamma = \int_0^t rdt;$$

so that $d\alpha = pdt$, &c. Then the above equations give

$$d\theta = -\cos\phi d\alpha + \sin\phi d\beta, \quad d\phi = d\gamma + \cot\theta(\sin\phi d\alpha + \cos\phi d\beta)$$
$$d\psi = -\operatorname{cosec}\theta(\sin\phi d\alpha + \cos\phi d\beta).$$

Here α, β, γ are the sums of the elementary angles described about the axes in the course of the motion; and no one would maintain that θ, ϕ, ψ are *functions* of α, β, γ, for the values of the latter variables at any time *do not determine* the values of the former. If therefore we choose to write such equations as

$$\frac{d\theta}{d\alpha} = -\cos\phi, \quad \frac{d\theta}{d\beta} = \sin\phi, \text{ \&c.},$$

we must admit that $\frac{d\theta}{d\alpha}, \frac{d\theta}{d\beta}$, &c. are not partial derived functions in the ordinary sense. At most, $\frac{d\theta}{d\alpha}$ is the derived function of *that function of* α *which* θ *would become* if β and γ were maintained invariable, *i. e.* if the motion were restricted to a rotation about the A-axis. Again, if we admit such symbols as $\frac{d^2\theta}{d\beta d\alpha}, \frac{d^2\theta}{d\alpha d\beta}$, we must interpret them as follows:

$$\frac{d^2\theta}{d\beta d\alpha} = \frac{d}{d\beta}\frac{d\theta}{d\alpha} = -\frac{d\cos\phi}{d\beta} = \sin\phi\frac{d\phi}{d\beta},$$

but $\frac{d\varphi}{d\beta}=\cot\theta\cos\varphi$, and therefore

$$\frac{d^2\theta}{d\beta d\alpha}=\cot\theta\sin\varphi\cos\varphi;$$

and in like manner we find the same value for $\frac{d^2\theta}{d\alpha d\beta}$, so that in *this particular case* the condition $\frac{d^2\theta}{d\alpha d\beta}=\frac{d^2\theta}{d\beta d\alpha}$ is verified.

But if we take $\frac{d^2\theta}{d\gamma d\alpha}$ and $\frac{d^2\theta}{d\alpha d\gamma}$ in the same way, we find *the former* $=\sin\varphi$ *and the latter* $=0$, so that in this case the condition *is not verified.* The geometrical meaning of this is obvious; analytically it is merely an instance of a general fact, pointed out by JACOBI; namely, that the effect of two successive *pseudo-differentiations* with respect to two independent variables, is not generally independent of the order of operations.

If V be the potential of another body, given in position, upon the body considered, then V is a function of θ, φ, ψ, and

$$dV=\frac{dV}{d\theta}d\theta+\frac{dV}{d\varphi}d\varphi+\frac{dV}{d\psi}d\psi;$$

and if we substitute for $d\theta$, $d\varphi$, $d\psi$ their values in terms of $d\alpha$, $d\beta$, $d\gamma$, we obtain an expression which we may call $Ld\alpha+Md\beta+Nd\gamma$, L, M, N being functions of θ, φ, ψ, $\frac{dV}{d\theta}$, $\frac{dV}{d\varphi}$, $\frac{dV}{d\psi}$, of which, as is well known, the mechanical meanings are the moments of the attraction of the second body about the three axes. Here again no one would maintain that V is a *function* of α, β, γ; and if, as is often done, we say $\frac{dV}{d\alpha}=L$, &c., the above remarks apply in all respects to these equations.

I should have thought it superfluous to dwell so much on these points if it had not appeared that writers on physical astronomy have in some instances either overlooked the distinction between *real* and *pseudo*-differentiation, or at least have failed to point it out to their readers. The only discussion of the subject which I have met with is that given by JACOBI, in the correspondence referred to.

It may be added, that in general investigations, where symbols such as $\frac{dV}{d\alpha}$, &c. may be used without defining the nature of V, or the precise meaning of α, β, &c., serious errors might be committed if it were assumed that the condition $\frac{d^2V}{d\alpha d\beta}=\frac{d^2V}{d\beta d\alpha}$ always subsisted.

APPENDIX C.

The theorems relating to the transformation of coordinates, given in Section VI., may be made more general, and in many cases more useful, as follows:—

If $x_1, x_2, \ldots x_n$ be the coordinates employed in the first statement of any dynamical problem, the differential equations are comprehended in the formula

$$\Sigma_i\left\{\left(\frac{dW}{dx'_i}\right)' - \frac{dW}{dx_i}\right\}\delta x_i = 0. \qquad \text{(D.)}$$

[If there be any forces, such as those arising from a resisting medium, which do not satisfy the natural conditions of integrability, then on the right-hand side of the formula (D.), instead of 0 we shall have an expression such as $\Sigma_i(X_i\delta x_i)$; but such terms are easily introduced and allowed for separately, and do not affect the following investigation. I shall therefore here assume that they do not exist.]

In the above formula, W is a given function of $x_1, \ldots x_n, x'_1, \ldots x'_n$, which may also explicitly contain t.

In Section VI. the only case contemplated was that in which $x_1, \ldots x_n$ are *independent* coordinates; in which case the formula (D.) is equivalent to n separate equations, since δx_1, &c. are wholly arbitrary and independent.

In practice, however, it is often more convenient to use, at first, a set of coordinates more in number than the independent variables of the problem, and therefore subject to equations of condition.

Let us assume then that the n coordinates $x_1, \ldots x_n$ are subject to r equations of condition,

$$L_1=0,\ L_2=0,\ \ldots\ L_r=0,$$

where L_1, &c. may explicitly contain t, besides the n variables x_1, &c.

If we introduce the n conjugate variables $y_1, \ldots y_n$ defined by the equations $y_i = \frac{dW}{dx'_i}$, and take Z a function of x_1, &c., y_1, &c. (with or without t), defined by the equation

$$Z = -[W] + [x'_1]y_1 + \ldots + [x'_n]y_n$$

(the brackets indicating that x'_1, &c. are expressed in terms of y_1, &c.), then it follows exactly as in art. 18 (Part I.), that the formula (D.) will be changed into the system

$$\left.\begin{array}{c} x'_i = \dfrac{dZ}{dy_i} \\ \Sigma_i\left(y'_i + \dfrac{dZ}{dx_i}\right)\delta x_i = 0 \end{array}\right\} \qquad \text{(E.)}$$

[In the most usual problems W is of the form T+U, where T is homogeneous and of the second degree in x'_1, &c., and U does not contain x'_1, &c. at all. In this case Z is only T−U expressed in terms of y_1, &c., instead of x'_1, &c. But T is *not necessarily* homogeneous; in fact it is not so in problems relating to motion *relative to the earth,* as affected by the earth's rotation.]

Let us now suppose that the system (E.) is to be transformed by the introduction of the m *independent* coordinates $\xi_1, \xi_2, \ldots \xi_m$, and of the new conjugate variables $\eta_1, \eta_2, \ldots \eta_m$; where $m=n-r$. And let it be required to investigate a theorem by means of which the transformation may be effected *without recurring to the original formula* (D.).

The definitions of the new coordinates ξ_1, &c. will furnish m equations (which may explicitly contain t) by means of which $\xi_1, \dots \xi_m$ may be expressed as functions of $x_1, \dots x_n$ (with or without t); and conversely, by means of these m equations, together with the r equations of condition $L_1=0$, &c., the n variables $x_1, x_2, \dots x_n$ may be expressed as functions of $\xi_1, \dots \xi_m$, with or without t. When $x_1, \dots x_n$ are so expressed, let them be represented by $(x_1), \dots (x_n)$. We shall have then

$$x'_i=\frac{d(x_i)}{dt}+\frac{d(x_i)}{d\xi_1}\xi'_1+\dots+\frac{d(x_i)}{d\xi_m}\xi'_m, \quad \dots \dots \dots \quad (x'.)$$

so that x'_1, &c. are expressible (and in only one way) as functions of ξ_1, &c., ξ'_1, &c.

If then the formula (D.) be transformed by expressing x_1, &c., x'_1, &c. in this manner, it becomes, as is well known,

$$\Sigma_i\left\{\left(\frac{d(\mathrm{W})}{d\xi'_i}\right)'-\frac{d(\mathrm{W})}{d\xi_i}\right\}\delta\xi_i=0,$$

where (W) represents the result of transforming W as above; and since $\delta\xi_1$, &c. are now independent, this formula breaks up into the m separate equations

$$\left(\frac{d(\mathrm{W})}{d\xi'_i}\right)'=\frac{d(\mathrm{W})}{d\xi_i}. \quad \dots \dots \dots \quad (\mathrm{F.})$$

Moreover, if we now define η_i by the equation $\frac{d(\mathrm{W})}{d\xi'_i}=\eta_i$, and put

$$\Psi=-(\mathrm{W})+(\xi'_1)\eta_1+\dots+(\xi'_m)\eta_m,$$

where (ξ'_1), &c. are expressed in terms of η_1, &c., we know already (art. 18.) that the system (F.) becomes

$$\xi'_i=\frac{d\Psi}{d\eta_i}, \quad \eta'_i=-\frac{d\Psi}{d\xi_i}. \quad \dots \dots \dots \quad (\mathrm{G.})$$

Now let P be a function of the m new variables $\xi_1, \dots \xi_m$, and of the n old variables $y_1, \dots y_n$ (with or without t), defined by the equation

$$\mathrm{P}=(x_1)y_1+(x_2)y_2+\dots+(x_n)y_n.$$

Since $\eta_i=\frac{d(\mathrm{W}}{d\xi'_i}$, and since (W) contains ξ'_i, &c., only through x'_i, &c., we have, observing that $\frac{d\mathrm{W}}{dx'_i}=y_i$,

$$\eta_i=y_1\frac{dx'_1}{d\xi'_i}+y_2\frac{dx'_2}{d\xi'_i}+\dots+y_n\frac{dx'_n}{d\xi'_i};$$

but from the equation $(x'.)$ we have $\quad \frac{dx'_j}{d\xi'_i}=\frac{d(x_j)}{d\xi_i}$,

consequently $\quad \eta_i=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\dots+y_n\frac{d(x_n)}{d\xi_i}$,

an expression evidently equivalent to $\frac{d\mathrm{P}}{d\xi_i}$. Thus η_i *may be defined by the equation*

$$\frac{d\mathrm{P}}{d\xi_i}=\eta_i. \quad \dots \dots \dots \quad (\eta.)$$

without recurring to the formula (D.). And since each equation of condition gives, if differentiated totally with respect to t, with the substitution of $\frac{dZ}{dy_i}$ for x'_i, an equation such as

$$0=\frac{dL}{dt}+\frac{dL}{dx_1}\frac{dZ}{dy_1}+..+\frac{dL}{dx_n}\frac{dZ}{dy_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (L.)$$

the m equations (η.) with the r equations L (in which last x_1, &c. must be expressed, after the differentiation, in terms of ξ, &c.), give n equations by means of which $y_1, \dots y_n$ may be expressed in terms of ξ_1, &c., η_1, &c., and can be so expressed *only in one way*.

Lastly, the value of Ψ (see equations (G.)), may be obtained as follows:—

Since $$\Psi=-(W)+(\xi'_1)\eta_1+\dots+(\xi'_m)\eta_m$$

and $$Z=-[W]+[x'_1]y_1+\dots+[x'_n]y_n,$$

and (W) is only [W] differently expressed, we have, without reference to modes of expression,

$$\Psi-Z=\Sigma_i(\xi'_i\eta_i)-\Sigma_i(x'_iy_i).$$

On the other hand, since P can contain t explicitly only through (x_1), &c., we have

$$\frac{dP}{dt}=\Sigma_i\left(y_i\frac{d(x_i)}{dt}\right),$$

and also $$\Sigma_i(x'_iy_i)=\Sigma_i\left(y_i\frac{d(x_i)}{dt}\right)+\Sigma_i\left(\frac{dP}{d\xi_i}\xi'_i\right);$$

hence, observing that $\frac{dP}{d\xi_i}=\eta_i$, we obtain

$$\Sigma_i(\xi'_i\eta_i)-\Sigma_i(x'_iy_i)=-\frac{dP}{dt};$$

and therefore, finally, $$\Psi=Z-\frac{dP}{dt},$$

so that $Z-\frac{dP}{dt}$ must become identical with Ψ, when expressed entirely in terms of the new variables.

These results may be stated in the form of the following *theorem*. The system (E.) is transformed into the system (G.) by the following substitutions:—

(1) $x_1, \dots x_n$ are expressed in terms of $\xi_1, \dots \xi_m$ by means of the m equations which define the latter variables, together with the $n-m$ equations of condition

$$L_1=0, \dots L_r=0 \ (r=n-m).$$

(2) $\eta_1, \dots \eta_m$ are defined by the m equations $\frac{dP}{d\xi_i}=\eta_i$; where the modulus of transformation P is given by the equation

$$P=(x_1)y_1+(x_2)y_2+\dots+(x_n)y_n,$$

(x_1), &c. being expressed in terms of ξ_1, &c., so that P is explicitly a function of $\xi_1, \dots \xi_m$, $y_1 \dots y_n$, with or without t.

(3) Ψ is defined by the equation $\Psi = Z - \frac{dP}{dt}$, in which (after the explicit differentiation of P with respect to t), x_1, &c., y_1, &c. are to be expressed in terms of the new variables. y_1, &c. are thus expressible by the help of the m equations $\frac{dP}{d\xi} = \eta_i$ and the $n-m$ equations $\frac{dL}{dt} + \Sigma_i \left(\frac{dL}{dx_i} \frac{dZ}{dy_i} \right) = 0$.

If (x_1), &c., do not contain t explicitly, then $\frac{dP}{dt} = 0$, and Ψ is obtained merely by expressing Z in terms of the new variables.

It may be observed that the whole of the above reasoning would apply to the case in which the new variables $\xi_1, \ldots \xi_m$ are more in number than the independent variables of the problem (or $m > n - r$), *with this exception*; that the m equations $\frac{dP}{d\xi_i} = \eta_i$, together with the r equations obtained by differentiating the equations of condition totally with respect to t, would be *more than sufficient* to express $y_1, \ldots y_n$ in terms of the new variables; consequently y_1, &c. might be so expressed in *different ways*, and therefore, although the *value* of Ψ obtained by the above rule would certainly be the same as that obtained by recurring to the original formula (D.), the *form* of Ψ might be different, and therefore the resulting formula erroneous.

There must doubtless exist some rule for choosing $n-m$ combinations of the equations of condition in such a way as to lead to the correct *forms* of $y_1, \ldots y_n$ as functions of the new variables; but I have not at present attempted to investigate it, and perhaps it would be hardly worth while. The theorem in the case in which the new coordinates are independent, may, I believe, be practically useful.

ERRATA IN PART I.

Art. 1. equation (4.), for dx read dx_i.

Art. 10. In paragraph preceding equation (26.) *omit* the words "not containing t explicitly."

Art. 18. equation (β), for y_i read y_i'.

Art. 19. equation (29.), for h_i read b_i.

Art. 24. second line after equation (L.), *for* "such as h, k" *read* "such as f, g."

Art. 30. The expressions equated to h, k, c, and the three terms in the left-hand column of the table of elements, should each be multiplied by m.

Art. 42. near the end, *for* "according as Θ is between o and π, or not" *read* "according as Θ is between π and 2π, or between o and π."